Lecture 11
Solving massive Thirring model by Bethe Ansatz: thermodynamic limit

On the last lecture we derived the Bethe equation for N pseudoparticles

$$p_0(\lambda_k)L + \sum_{l=1}^{N} \Phi(\lambda_k - \lambda_l) = 2\pi n_k, \qquad n_k \in \mathbb{Z} + \frac{\delta}{2}. \tag{1}$$

BetheEquatio

Today we will use it to describe the true vacuum of the system and excitations above the vacuum. The Bethe equation is a system of nonlinear algebraic equations so that it is probable impossible to solve it analytically. For the Thirring model there is no need to solve them for arbitrary finite N, since we are only interested in the limit $N \to \infty$. In principle, this limit is achieved by sending to infinity the cut-off scale Λ, and we may consider arbitrary values of L. In fact, here we will only discuss the case of large enough L, namely $L \gg m^{-1}$, where m is the physical mass of excitations.

When $g = 0$ the vacuum had the following structure. The parameters λ_k lied on the axis $\mathbb{R} + \mathrm{i}\pi$, so that $\lambda_k = \mathrm{i}\pi + \beta_k$. It corresponded to the negative energy 'excitations'. Besides, the negative energy band was full, so that we could write $n_k = -k + \text{const}$. In the limit $L \to \infty$ the roots β_k became dense. Note that in this dense case the answer is not sensible (in the leading approximation) to the periodicity condition (NS or R). Suppose that this structure is preserved for small enough values of g. Then for the *vacuum solution* we have

$$p_0(\beta_k)L = \sum_{l=1}^{N} \Phi(\beta_k - \beta_l) + 2\pi k + \text{const}.$$

Let us take a difference of two these equations with neighboring values of k:

$$p_0(\beta_{k+1}) - p_0(\beta_k) = \frac{1}{L}\sum_{l=1}^{N}(\Phi(\beta_{k+1} - \beta_l) - \Phi(\beta_k - \beta_l) + \frac{2\pi}{L}.$$

Define the *density of roots* for the vacuum solution:

$$\rho(\beta_k) = \frac{1}{L(\beta_{k+1} - \beta_k)}.$$

If roots are dense, we may write $f(\lambda_{k+1}) - f(\lambda_k) = f'(\lambda_k)(\lambda_{k+1} - \lambda_k)$ for any slowly changing function f. Hence,

$$p'(\beta_k) = \frac{1}{L}\sum_{l=1}^{N} \Phi'(\beta_k - \beta_l) + 2\pi\rho(\beta_k).$$

Finally substitute the sum by and integral with the measure $dl = L\rho(\beta_l)\, d\beta_l$:

$$p'(\beta) = 2\pi\rho(\beta) + \int_{-\Theta}^{\Theta} d\gamma\, \Phi'(\beta - \gamma)\rho(\gamma) \quad \text{for } -\Theta \le \beta \le \Theta. \tag{2}$$

BetheEquatio

The parameter β_0 is related to the cut-off parameter Λ:

$$\Lambda = \frac{m_0}{2}\mathrm{e}^{\Theta}. \tag{3}$$

Lambda-Theta-

On the other hand, it is related to the spacial density of pseudoparticles in the vacuum state:

$$\frac{N}{L} = \int_{-\Theta}^{\Theta} d\beta\, \rho(\beta). \tag{4}$$

N-Theta-rel

We do not intend to solve the equation (2) since it only gives the vacuum energy, which we are not interested in. It can be shown that

$$\rho(\beta) \propto \exp\frac{\beta}{1+g}, \tag{5}$$

rho-fin

but it will not help us. We will be interested in excitations. There is a large zoo of excitations in this theory, but we will consider the simplest kind of them: holes. Consider the state in which $\lambda_k = \mathrm{i}\pi + \beta_k$, $\beta_k \in \mathbb{R}$, as before, but the values n_k do not form a solid segment, but have some omissions.

Return for a moment to the general equation (1). Let $\{\lambda_k\}$ be a given solution, and $\{n_k\}$ is the corresponding set of integers (or half-integers). Then define the function $\lambda(n)$ as a solution to the equation

$$\pi(\lambda(n))L + \sum_{l=1}^{N} \Phi(\lambda(n) - \lambda_l) = 2\pi n. \tag{6}$$

lambda(n)-de

If $n = n_k$ the value of the function coincides with the corresponding root: $\lambda(n_k) = \lambda_k$.

Return to our dense states. If we order the set of n_k in such a way that $n_{k+1} < n_k$, we may define the *density of states* $\rho(\beta)$ and the *density of roots* $\rho_r(\beta)$:

$$\rho(\beta(n)) = \frac{1}{L(\beta(n+1) - \beta(n))}, \qquad \rho_r(\beta(n_k)) = \frac{1}{L(\beta_{k+1} - \beta_k)}. \tag{7}$$

rho-rhor-def

We also may define the *density of holes*

$$\rho_h(\beta) = \rho(\beta) - \rho_r(\beta). \tag{8}$$

rhoh-def

It is easy to write the integral equation for such a state

$$p'(\beta) = 2\pi\rho(\beta) + \int_{-\Theta}^{\Theta} d\gamma\, \Phi'(\beta - \gamma)(\rho(\gamma) - \rho_h(\gamma)) \quad \text{for } -\Theta \le \beta \le \Theta. \tag{9}$$

BetheAnsatz-

It is important that $\rho(\beta)$ is the unknown, while $\rho_h(\beta)$ is a free parameter. We may choose any function $\rho_h(\beta)$ if only the solution will satisfy $\rho(\beta) \ge \rho_h(\beta)$. For example, the choice $\rho_h(\beta) = L^{-1}\delta(\beta - \beta_0)$ corresponds to the state with a single hole of rapidity β_0.

Let $\rho_0(\beta)$ be the vacuum density of states, i.e. the solution to the equation (2) and $\rho(\beta) = \rho_0(\beta) + \delta\rho(\beta)$ be the solution to the equation (9) with a given density of holes $\rho_h(\beta)$. By taking the difference of these two equations we obtain

$$2\pi\delta\rho(\beta) + \int_{-\Theta}^{\Theta} d\gamma\, \Phi'(\beta - \gamma)\delta\rho(\gamma) = \int_{-\Theta}^{\Theta} d\gamma\, \Phi'(\beta - \gamma)\rho_h(\gamma). \tag{10}$$

deltarho-eq

Note that $\delta\rho(\beta)$ is not supposed to be small.

Now let us solve the equation. It turns out that it is sufficient to solve it for $\Theta = \infty$. In this case we may apply the Fourier method. Let

$$\tilde\Phi'(\omega) = \int_{-\infty}^{\infty} d\beta\, \Phi'(\beta) \mathrm{e}^{\mathrm{i}\omega\beta} = -2\pi \frac{\operatorname{sh} \pi g\omega}{\operatorname{sh} \pi\omega}, \tag{11}$$

Phi'-Fourier

$$\delta\tilde\rho(\omega) = \int_{-\infty}^{\infty} d\beta\, \delta\rho(\beta) \mathrm{e}^{\mathrm{i}\omega\beta}, \qquad \tilde\rho_h(\omega) = \int_{-\infty}^{\infty} d\beta\, \rho_h(\beta) \mathrm{e}^{\mathrm{i}\omega\beta}. \tag{12}$$

rho-Fourier

After the Fourier transform equation (10) reads

$$2\pi\delta\tilde\rho(\omega) + \tilde\Phi'(\omega)\delta\tilde\rho(\omega) = \tilde\Phi'(\omega)\tilde\rho_h(\omega). \tag{13}$$

deltarho-eq-

It solution is given by

$$\delta\tilde\rho(\omega) = -\frac{\operatorname{sh} \pi g\omega}{2 \operatorname{sh} \frac{\pi(1-g)\omega}{2} \operatorname{ch} \frac{\pi(1+g)\omega}{2}} \tilde\rho_h(\omega). \tag{14}$$

deltarho-sol

Suppose that $\tilde\rho_h(\omega)$ does not have poles in a large enough strip around the real axis. In particular, for a finite number of holes this function has no poles at all. Compute the energy (comparing to the vacuum energy) and momentum of the state:

$$\begin{aligned} E &= L \int_{-\Theta}^{\Theta} d\beta\, \epsilon_0(\beta)(\rho_h(\beta) - \delta\rho(\beta)), \\ P &= L \int_{-\Theta}^{\Theta} d\beta\, p_0(\beta)(\rho_h(\beta) - \delta\rho(\beta)) \operatorname{sh} \beta. \end{aligned} \tag{15}$$

energy-momen

Consider the case $g<0$. In this case these integrals are convergent as $\Theta\to\infty$, and we easily obtain that for $\Theta=\infty$

$$E+P=m_0L(\tilde\rho_h(\mathrm{i})-\delta\tilde\rho(\mathrm{i}))=0,\qquad E-P=m_0L(\tilde\rho_h(-\mathrm{i})-\delta\tilde\rho(-\mathrm{i}))=0.$$

In fact, surely, the energy and momentum are nonzero due to finiteness of Θ. It can be shown that

$$E=L\int_{-\Theta}^{\Theta}d\beta\,\rho_h(\beta)\epsilon\left(\frac{\beta}{1+g}\right),\qquad P=L\int_{-\Theta}^{\Theta}d\beta\,\rho_h(\beta)p\left(\frac{\beta}{1+g}\right),\tag{16}$$ EP-fin

where $\epsilon(\theta),p(\theta)$ are energy and momentum of physical excitations:

$$\epsilon(\theta)=m\,\mathrm{ch}\,\theta,\qquad p(\theta)=m\,\mathrm{sh}\,\theta,\tag{17}$$ ep-phys

$$m=\frac{m_0}{g}\,\mathrm{ctg}\left(\frac{\pi}{2}\frac{1-g}{1+g}\right)\mathrm{e}^{\frac{g}{1+g}\Theta}\sim m_0\left(\frac{m_0}{\Lambda}\right)^{-\frac{g}{1+g}}.\tag{18}$$ m-phys

Comparing with the case $g=0$ we conclude that this excitation is an antifermion[1] We see that the rapidity is renormalized in consistency with the density of states (5).

Nevertheless, here we have a problem. We have concluded in (18) that the physical mass is related to the bare mass as $m\sim m_0(m_0/\Lambda)^{g/(1+g)}\sim m_0(m_0r)^{g/(1+g)}$. But this contradicts the relation obtained by the boson–fermion correspondence:

$$m\sim m_0(m_0r)^{\frac{1}{2(1-\beta^2)}-1}\sim m_0(m_0r)^{-\frac{g}{1+2g}}.$$

The formula (18) would be obtained if we assumed $g=1-2\beta^2$ instead of $g=(2\beta^2)^{-1}-1$. We may conjecture that the coupling constant g in the Bethe Ansatz, being based on the anticommutation relation for fermions, differs with that of obtained from operator product expansion due renormalization of the fermion. If we denote the coupling constant in Bethe Ansatz by g_{BA} and that used in the boson–fermion correspondence by g_{OPT}, we get

$$g_{\mathrm{BA}}=\frac{g_{\mathrm{OPE}}}{1+g_{\mathrm{OPE}}}.\tag{19}$$ g-BA-OPE

In fact, after assuming this relation all results of the Bethe Ansatz and of the conformal perturbation theory become consistent.

Can we say anything about interaction of the antifermions? Yes. To extract this information, consider an auxiliary model of spinless fermions of mass m with the wave function of the form

$$\chi_{\theta_1\dots\theta_N}(x_1,\dots,x_N)=\sum_{\tau\in S^N}(-1)^{\sigma\tau}A_\tau\prod_{k=1}^N\mathrm{e}^{\mathrm{i}mx_{\sigma_k}\,\mathrm{sh}\,\theta_{\tau_k}},\quad \text{if } x_{\sigma_1}<\dots<x_{\sigma_N}.$$

The constants A_τ are related as

$$A_{\dots ij\dots}=S^{-1}(\theta_i-\theta_j)A_{\dots ji\dots},\tag{20}$$ A=SA

where $S(\theta)$ is a given function, which play a role of the S matrix of two particles. In fact, this model is an approximation to system of physical fermions in a strong external field, which suppress their antifermions. As we will see below, wave functions with such long-distance asymptotics are characteristic for integrable systems.

Imposing the periodicity condition produces the Bethe equation

$$\mathrm{e}^{\mathrm{i}p(\lambda_k)L}=\pm\prod_{\substack{l=1\\(l\ne k)}}^N S(\theta_k-\theta_l).\tag{21}$$ BE-S

Let

$$S(\theta)=\mathrm{e}^{\mathrm{i}\Psi(\theta)}.\tag{22}$$ S-Psi

[1]Note that if we try to obtain the fermion as a pseudoparticle with real λ, we will fail. For indefinitely small though nonzero g the pseudoparticle repulse states in the vicinity of $\lambda+\mathrm{i}\pi$, and we obtain a fermion-antifermion bound state instead. The construction of the fermion is more involved.

In the limit of dense roots we obtain

$$p'(\theta) + \int_{-\infty}^{\infty} d\vartheta \Psi'(\theta - \vartheta)\rho_p(\vartheta) = 2\pi\rho_s(\vartheta). \tag{23}$$ BetheEquation

Let $\delta\rho_s(\theta) = \rho_s(\theta) - \frac{1}{2\pi}p'(\theta)$. Then we get

$$\delta\rho_s(\vartheta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\vartheta \Psi'(\theta - \vartheta)\rho_p(\vartheta). \tag{24}$$ deltarhos-S

After applying the Fourier transform

$$\tilde\rho_p(t) = \int_{-\infty}^{\infty} d\theta\, \rho_p(\theta) \mathrm{e}^{\mathrm{i}t\theta} \quad \text{etc.}$$

we obtain

$$\delta\tilde\rho_s(t) = \frac{1}{2\pi}\tilde\Psi'(t)\tilde\rho_p(t).$$

If we assume that these particles correspond to holes in the Bethe Ansatz solution to the Thirring model, then the density of such particles $\rho_p(\theta)$ is proportional to the density of holes $\rho_h(\beta)$, while the density of states $\rho_s(\theta)$ is proportional to the density of roots $\rho(\beta)$:

$$\rho_p(\theta) = (1+g)\rho_h((1+g)\theta), \qquad \rho_s(\theta) = (1+g)\rho((1+g)\theta). \tag{25}$$ rho-rel

In terms of Fourier components we have

$$\tilde\rho_p(t) = \tilde\rho_h\left(\frac{t}{1+g}\right), \qquad \delta\tilde\rho_s(t) = \delta\tilde\rho\left(\frac{t}{1+g}\right). \tag{26}$$ tilderho-rel

Hence,

$$\delta\tilde\rho\left(\frac{t}{1+g}\right) = \frac{1}{2\pi}\tilde\Psi'(t)\tilde\rho_h\left(\frac{t}{1+g}\right)$$

By substituting (14) we obtain

$$\tilde\Psi'(t) = -2\pi\frac{\operatorname{sh}\pi\frac{\pi g t}{1+g}}{2\operatorname{sh}\frac{\pi(1-g)t}{(1+g)}\operatorname{ch}\frac{\pi t}{2}}. \tag{27}$$ Psi'-solution

Substituting g by $1-2\beta^2$ and using the parametrization

$$\beta^2 = \frac{p}{p+1},$$

we obtain

$$S(\theta) = \exp\left(-\int\frac{dt}{t}\frac{\operatorname{sh}\frac{\pi t}{2}\operatorname{sh}\frac{\pi(p-1)t}{2}}{\operatorname{sh}\pi t\operatorname{sh}\frac{\pi p t}{2}}\mathrm{e}^{-\mathrm{i}\theta t}\right) = \exp\left(2\mathrm{i}\int\frac{dt}{t}\frac{\operatorname{sh}\frac{\pi t}{2}\operatorname{sh}\frac{\pi(p-1)t}{2}}{\operatorname{sh}\pi t\operatorname{sh}\frac{\pi p t}{2}}\sin\theta t\right). \tag{28}$$ S-fin

Note that it is the S matrix, which describes the scattering of two antifermions into two antifermions $S(\theta)^{--}_{--}$. Due to the charge symmetry of the model it coincides with the scattering of two fermions int two fermions $S(\theta)^{++}_{--}$. Due to the crossing symmetry of amplitudes it also provides the matrix elements $S(\theta)^{+-}_{+-}$ and $S(\theta)^{-+}_{-+}$:

$$S(\theta)^{++}_{++} = S(\theta)^{--}_{--} = S(\theta), \qquad S(\theta)^{+-}_{+-} = S(\theta)^{-+}_{-+} = S(\mathrm{i}\pi - \theta) = S(\theta)\frac{\operatorname{sh}\frac{\theta}{p}}{\operatorname{sh}\frac{\mathrm{i}\pi-\theta}{p}}. \tag{29}$$ 4S-fin

To obtain the whole S matrix we have to learn more about general properties of integrable systems.

Bibliography

[1] V. E. Korepin, N. M. Bogoliubov, A. G. Izergin, Quantum Inverse Scattering Method and Correlation Functions, Cambridge, 1997.

Problems

1. Derive (16)–(18).
2. Calculate the change of the number of states in the interval $[-\Theta, \Theta]$ under the influence of holes:

$$\delta n = \int_{-\Theta}^{\Theta} d\beta \, \delta\rho(\beta).$$