Lecture 7
Operator product expansions, crossing symmetry and conformal blocks

Return for a while to a general field theory. Let $\{\mathcal{O}_I\}$ be a complete system of mutually local linearly independent operators. The completeness will be understood in the sense of operator product expansions (OPE) in the Euclidean theory:[1]

$$\mathcal{O}_I(x')\mathcal{O}_J(x) = \sum_K D_{IJ}^K(x',x)\mathcal{O}_K(x). \tag{1}$$

We will assume the series

$$\sum_K D_{IJ}^K(x',x)\langle\mathcal{O}_K(x)\mathcal{O}_{M_1}(x_1)\ldots\mathcal{O}_{M_N}(x_N)\rangle$$

converge for small enough $|x'-x|$. The functions $D_{IJ}^K(x',x)$ are called *structure functions*. Consider the product of three operators $\mathcal{O}_{I_3}(x_3)\mathcal{O}_{I_2}(x_2)\mathcal{O}_{I_1}(x_1)$. We may do it in several ways. We will consider two of them. We may first fuse $\mathcal{O}_{I_2}$ with $\mathcal{O}_{I_1}$ and then the result with $\mathcal{O}_{I_3}$. Alternatively, we may first fuse $\mathcal{O}_{I_3}$ with $\mathcal{O}_{I_2}$ and then with $\mathcal{O}_{I_1}$. Let us decompose about the point x_2. In the first manner we have

$$\begin{aligned}
&\overbrace{\mathcal{O}_{I_3}(x_3)\,\overbrace{\mathcal{O}_{I_2}(x_2)\mathcal{O}_{I_1}(x_1)}} = (-)^{(I_1,I_2)}\,\overbrace{\mathcal{O}_{I_3}(x_3)\,\overbrace{\mathcal{O}_{I_1}(x_1)\mathcal{O}_{I_2}(x_2)}} \\
&\quad= (-)^{(I_1,I_2)}\sum_J D_{I_1I_2}^J(x_1,x_2)\mathcal{O}_{I_3}(x_3)\mathcal{O}_J(x_2) = (-)^{(I_1,I_2)}\sum_{J,I_4} D_{I_3J}^{I_4}(x_3,x_2)D_{I_1I_2}^J(x_1,x_2)\mathcal{O}_{I_4}(x_2).
\end{aligned}$$

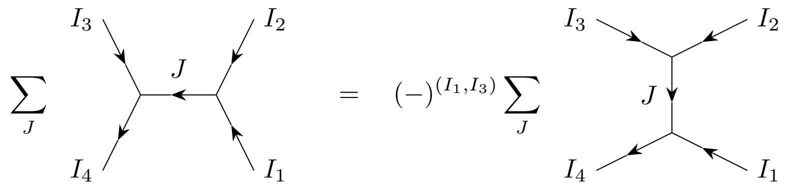
Here $(-)^{(I,J)} = (-1)^{4s_Is_J}$ (s_I being the spin of $\mathcal{O}_I$) is -1, if both operators are fermionic and 1 otherwise. Alternatively, we have

$$\begin{aligned}
&\overbrace{\overbrace{\mathcal{O}_{I_3}(x_3)\mathcal{O}_{I_2}(x_2)}\,\mathcal{O}_{I_1}(x_1)} = (-)^{(I_1,I_2)+(I_1,I_3)}\,\overbrace{\mathcal{O}_{I_1}(x_1)\,\overbrace{\mathcal{O}_{I_3}(x_3)\mathcal{O}_{I_2}(x_2)}} \\
&= (-)^{(I_1,I_2)+(I_1,I_3)}\sum_J D_{I_3I_2}^J(x_3,x_2)\mathcal{O}_{I_1}(x_1)\mathcal{O}_J(x_2) = (-)^{(I_1,I_2)+(I_1,I_3)}\sum_{J,I_4} D_{I_1J}^{I_4}(x_1,x_2)D_{I_3I_2}^J(x_3,x_2)\mathcal{O}_{I_4}(x_2).
\end{aligned}$$

Comparison of these two expansions provides the *crossing symmetry* equation

$$\sum_J D_{I_3J}^{I_4}(x_3,x_2)D_{I_1I_2}^J(x_1,x_2) = (-)^{(I_1,I_3)}\sum_J D_{I_1J}^{I_4}(x_1,x_2)D_{I_3I_2}^J(x_3,x_2), \tag{2}$$

which can be illustrated graphically as



The correlation function are expressed in terms of operator product expansions and vacuum expectation values $\langle\mathcal{O}_I(x)\rangle$. For example, the two-, three- and four-point correlation functions read

$$\langle\mathcal{O}_{I_2}(x_2)\mathcal{O}_{I_1}(x_1)\rangle = \sum_J D_{I_1I_2}^J(x_2,x_1)\langle\mathcal{O}_J(x_1)\rangle, \tag{3}$$

$$\langle\mathcal{O}_{I_3}(x_3)\mathcal{O}_{I_2}(x_2)\mathcal{O}_{I_1}(x_1)\rangle = \sum_{JK} D_{I_3J}^K(x_3,x_1)D_{I_2I_1}^J(x_2,x_1)\langle\mathcal{O}_K(x_1)\rangle, \tag{4}$$

$$\langle\mathcal{O}_{I_4}(x_4)\mathcal{O}_{I_3}(x_3)\mathcal{O}_{I_2}(x_2)\mathcal{O}_{I_1}(x_1)\rangle = \sum_{JKL} D_{I_4K}^L(x_4,x_1)D_{I_3J}^K(x_3,x_1)D_{I_2I_1}^J(x_2,x_1)\langle\mathcal{O}_L(x_1)\rangle. \tag{5}$$

[1] In the Minkowski space we shall consider their analytic continuations.

Now consider a conformal field theory with a set of local operators $\{\mathcal{O}_I\}$ of fixed conformal dimensions $(\Delta_I, \bar{\Delta}_I)$. Stress that the operators $\mathcal{O}_I$ are not necessarily primary. They are simply eigenvectors of the operators $L_0, \bar{L}_0$. From the translation, rotation and dilatation invariance we immediately conclude that

$$D^K_{IJ}(x', x) = C^K_{IJ}(z' - z)^{\Delta_K - \Delta_I - \Delta_J}(\bar{z}' - \bar{z})^{\bar{\Delta}_K - \bar{\Delta}_I - \bar{\Delta}_J}. \tag{6}$$

Another simplification of CFT is that (see Problem 2 to the last lecture)

$$\langle \mathcal{O}_I(x) \rangle = 0, \quad \text{if } \Delta_I \neq 0 \text{ or } \bar{\Delta}_I \neq 0, \tag{7}$$

for any field $\mathcal{O}_I$, and

$$\langle \mathcal{O}_I(x')\mathcal{O}_J(x) \rangle = 0, \quad \text{if } \Delta_I \neq \Delta_J \text{ or } \bar{\Delta}_I \neq \bar{\Delta}_J, \tag{8}$$

and if the fields $\mathcal{O}_I(x), \mathcal{O}_J(x)$ are Möbius invariant.

Consider for simplicity a model of CFT that possesses the only field of zero dimension $\mathcal{O}_0 = 1$. In this case the formulas (3)–(5) simplify:

$$\langle \mathcal{O}_{I_2}(x_2)\mathcal{O}_{I_1}(x_1) \rangle = C^0_{I_2 I_2}(z' - z)^{-\Delta_{I_1} - \Delta_{I_2}}(\bar{z}' - \bar{z})^{-\bar{\Delta}_{I_1} - \bar{\Delta}_{I_2}}, \tag{9}$$

$$\begin{aligned}\langle \mathcal{O}_{I_3}(x_3)\mathcal{O}_{I_2}(x_2)\mathcal{O}_{I_1}(x_1) \rangle &= \sum_J C^0_{I_3 J} C^J_{I_2 I_1}(z_3 - z_1)^{-\Delta_{I_3} - \Delta_J}(z_2 - z_1)^{\Delta_J - \Delta_{I_2} - \Delta_{I_1}} \\ &\quad \times (\bar{z}_3 - \bar{z}_1)^{-\bar{\Delta}_{I_3} - \bar{\Delta}_J}(\bar{z}_2 - \bar{z}_1)^{\bar{\Delta}_J - \bar{\Delta}_{I_2} - \bar{\Delta}_{I_1}},\end{aligned} \tag{10}$$

$$\begin{aligned}\langle \mathcal{O}_{I_4}(x_4)\mathcal{O}_{I_3}(x_3)\mathcal{O}_{I_2}(x_2)\mathcal{O}_{I_1}(x_1) \rangle &= \sum_{JK} C^0_{I_4 K} C^K_{I_3 J} C^J_{I_2 I_1} \\ &\quad \times (z_4 - z_1)^{-\Delta_{I_4} - \Delta_K}(z_3 - z_1)^{\Delta_K - \Delta_{I_3} - \Delta_J}(z_2 - z_1)^{\Delta_J - \Delta_{I_2} - \Delta_{I_1}} \\ &\quad \times (\bar{z}_4 - \bar{z}_1)^{-\bar{\Delta}_{I_4} - \bar{\Delta}_K}(\bar{z}_3 - \bar{z}_1)^{\bar{\Delta}_K - \bar{\Delta}_{I_3} - \bar{\Delta}_J}(\bar{z}_2 - \bar{z}_1)^{\bar{\Delta}_J - \bar{\Delta}_{I_2} - \bar{\Delta}_{I_1}}.\end{aligned} \tag{11}$$

Now let us recall two important facts. First, the operators in CFT are classified by highest weight representations of the Virasoro algebra:

$$\mathcal{O}_I(x) = (L_{-k_1} \cdots L_{-k_r} \bar{L}_{-l_1} \cdots \bar{L}_{-l_s} \Phi_i)(x), \qquad k_1 \leq \cdots \leq k_r, \ l_1 \leq \cdots \leq l_s, \qquad I = (i, \vec{k}, \vec{l}). \tag{12}$$

Such set is generally linearly dependent, since the Verma modules are, in general, degenerated and must be reduced to irreducible representations by factorization over submodules. Nevertheless, keeping in mind this fact, we continue.

For the sake of simplicity we will write $\Phi_1, \Phi_2, \ldots$ instead of $\Phi_{i_1}, \Phi_{i_2}, \ldots$ from now on. The same simplified notation will be used for Δ_i, $C^k_{ij} = C^{(k,\varnothing,\varnothing)}_{(i,\varnothing,\varnothing)(j,\varnothing,\varnothing)}$.

Second, the operators $L_{-k}, \bar{L}_{-k}$ act on correlation functions as differential operators. Hence, if we only find the correlation functions of primary operators, we are able to compute any correlation function. Thus consider the functions

$$\langle \Phi_2(x_2)\Phi_1(x_1) \rangle = \frac{g_{21}}{(z_2 - z_1)^{2\Delta_1}(\bar{z}_2 - \bar{z}_1)^{2\bar{\Delta}_1}}, \qquad g_{12} = \begin{cases} C^0_{21}, & \text{if } \Delta_1 = \Delta_2, \ \bar{\Delta}_1 = \bar{\Delta}_2; \\ 0 & \text{otherwise.} \end{cases} \tag{13}$$

Here g_{21} is a matrix, which plays the role of metrics on the space of primary fields of a given dimension. In principle, by taking appropriate linear transformations we can make it any fixed non-degenerate matrix, e.g. the unit matrix. But for some time we retain it for generality. Nevertheless, we must remember that this matrix is *given and fixed*, it is not something we mean to compute.

Now consider three-point functions. Now we have an infinite sum

$$\begin{aligned}\langle \Phi_{i_3}(x_3)\Phi_{i_2}(x_2)\Phi_{i_1}(x_1) \rangle = \sum_{j,\vec{k},\vec{l}} C^0_{i_3(j,\vec{k},\vec{l})} C^{(j,\vec{k},\vec{l})}_{i_2 i_1}(z_3 - z_1)^{-2\Delta_{i_3}}(z_2 - z_1)^{\Delta_{i_3} - \Delta_{i_1} - \Delta_{i_2}} \left(\frac{z_2 - z_1}{z_3 - z_1}\right)^{\|\vec{k}\|} \\ \times (\bar{z}_3 - \bar{z}_1)^{-2\bar{\Delta}_{i_3}}(\bar{z}_2 - \bar{z}_1)^{\bar{\Delta}_{i_3} - \bar{\Delta}_{i_1} - \bar{\Delta}_{i_2}} \left(\frac{\bar{z}_2 - \bar{z}_1}{\bar{z}_3 - \bar{z}_1}\right)^{\|\vec{l}\|}.\end{aligned}$$

Here $\|\vec{k}\|=\sum_i k_i$. Now consider the limit $x_3\to\infty$ or, more precisely, $|z_3-z_1|\gg|z_2-z_1|$. In this limit the leading contribution comes from the primary intermediate operator: $\|\vec{k}\|=\|\vec{l}\|=0$. Hence,

$$\begin{aligned}\langle\Phi_3(x_3)\Phi_2(x_2)\Phi_1(x_1)\rangle=\sum_j g_{i_3j}C^j_{i_2i_1}(z_3-z_1)^{-2\Delta_3}(z_2-z_1)^{\Delta_3-\Delta_1-\Delta_2}\\ \times(\bar z_3-\bar z_1)^{-2\bar\Delta_3}(\bar z_2-\bar z_1)^{\bar\Delta_3-\bar\Delta_1-\bar\Delta_2}\quad\text{as } |z_3-z_1|\gg|z_2-z_1|.\end{aligned}$$

By comparing with the formula from Problem 3 of the last lecture, we obtain

$$\begin{aligned}\langle\Phi_3(x_3)\Phi_2(x_2)\Phi_1(x_1)\rangle=C_{321}(z_3-z_2)^{\Delta_1-\Delta_2-\Delta_3}(z_3-z_1)^{\Delta_2-\Delta_1-\Delta_3}(z_2-z_1)^{\Delta_3-\Delta_1-\Delta_2}\\ \times(\bar z_3-\bar z_2)^{\bar\Delta_1-\bar\Delta_2-\bar\Delta_3}(\bar z_3-\bar z_1)^{\bar\Delta_2-\bar\Delta_1-\bar\Delta_3}(\bar z_2-\bar z_1)^{\bar\Delta_3-\bar\Delta_1-\bar\Delta_2},\end{aligned}\tag{14}$$

where

$$C_{321}=\sum_j g_{3j}C^j_{21}.\tag{15}$$

We see that the knowledge of all structure constants related to primary operators in a conformal field theory is equivalent to the knowledge of all three-point correlation functions. This provides an evident symmetry property

$$C_{321}=(-1)^{(2,3)}C_{231}=(-1)^{(1,2)}C_{312}.\tag{16}$$

Now turn to the most complicated case: the four-point function. Without loss of generality we will consider the limit $x_4\to\infty$ and fix $z_3=1$, $z_1=0$. Indeed, consider the Möbius transformation

$$w=f(z)\equiv\frac{z_3-z_4}{z_3-z_1}\frac{z-z_1}{z-z_4}.\tag{17}$$

This transformation maps points $z=z_1,z_2,z_4$ to $w=0,1,\infty$. If we define

$$\langle\Phi_i(\infty)X\rangle=\langle\Phi_i|X|0\rangle=\lim_{z,\bar z\to\infty}z^{2\Delta_i}\bar z^{2\bar\Delta_i}\langle\Phi_i(z,\bar z)X\rangle,\tag{18}$$

we obtain

$$\begin{aligned}\langle\Phi_4(x_4)\Phi_3(x_3)\Phi_2(x_2)\Phi_1(x_1)\rangle&=K(\vec z,\vec\Delta)K(\vec{\bar z},\vec{\bar\Delta})\langle\Phi_4(\infty)\Phi_3(1)\Phi_2(w,\bar w)\Phi_1(0)\rangle,\\ K(\vec z,\vec\Delta)&=\frac{(z_1-z_3)^{\sum\Delta_i}(z_4-z_2)^{2\Delta_2}}{(z_4-z_1)^{\sum\Delta_i-2\Delta_1}(z_4-z_3)^{\sum\Delta_i-2\Delta_3}}.\end{aligned}\tag{19}$$

For brevity we wrote here $\Phi(0)$ instead of $\Phi(0,0)$ etc.

From (11) we obtain

$$\langle\Phi_4(\infty)\Phi_3(1)\Phi_2(z,\bar z)\Phi_1(0)\rangle=\sum_{J\equiv(j,\vec k,\vec l)}C_{43J}C^J_{21}z^{\Delta_j-\Delta_1-\Delta_2+\|\vec k\|}\bar z^{\bar\Delta_j-\bar\Delta_1-\bar\Delta_2+\|\vec l\|}.$$

Now I want to show you that the structure constants C_{43J} and C^J_{21} factorize as follows

$$\begin{aligned}C_{43J}&=C_{43j}\gamma^{\vec k}_{43j}\bar\gamma^{\vec l}_{43j},\\ C^J_{21}&=C^j_{21}\beta^{\vec k}_{j21}\bar\beta^{\vec l}_{j21},\end{aligned}\tag{20}$$

so that the constants $\gamma,\bar\gamma,\beta,\bar\beta$ are uniquely defined by the conformal dimensions of the corresponding primary fields. For the constants C_{43J} it is rather easy. Indeed,

$$\begin{aligned}C_{43J}&=\langle\Phi_4|\Phi_3(z,\bar z)L_{-k_1}\cdots L_{-k_r}\bar L_{-l_1}\cdots\bar L_{-k_s}|\Phi_j\rangle|_{z=\bar z=1}\\ &=(-1)^{r+s}C_{43j}(\mathcal{L}_{-k_1}\cdots\mathcal{L}_{-k_r}z^{\Delta_4-\Delta_3-\Delta_j})(\bar{\mathcal{L}}_{-l_1}\cdots\bar{\mathcal{L}}_{-l_s}\bar z^{\bar\Delta_4-\bar\Delta_3-\bar\Delta_j})\Big|_{z=\bar z=1},\end{aligned}$$

where

$$\mathcal{L}_k=(k+1)z^k\Delta_3+z^{k+1}\partial,\qquad\bar{\mathcal{L}}_k=(k+1)z^k\bar\Delta_3+\bar z^{k+1}\partial$$

Evidently,

$$\gamma_{43j}^{\vec{k}} = (-1)^r \; \mathcal{L}_{-k_1} \cdots \mathcal{L}_{-k_r} z^{\Delta_4-\Delta_3-\Delta_j}\Big|_{z=1},$$
$$\bar{\gamma}_{43j}^{\vec{k}} = (-1)^s \; \bar{\mathcal{L}}_{-k_1} \cdots \bar{\mathcal{L}}_{-k_r} \bar{z}^{\bar{\Delta}_4-\bar{\Delta}_3-\bar{\Delta}_j}\Big|_{\bar{z}=1}.$$

The situation with C_{21}^J is more complicated. Consider the product $\Phi_2(z,\bar{z})|\Phi_1\rangle$ and act on it by L_k with $k>0$:

$$L_k\Phi_2(z,\bar{z})|\Phi_1\rangle = ((k+1)z^k\Delta_2 + z^{k+1}\partial)\Phi_2(z,\bar{z})|\Phi_1\rangle.$$

Now apply the operator product expansion to both sides:

$$\sum_J C_{21}^J z^{\Delta_j+\|\vec{k}\|-\Delta_1-\Delta_2}\bar{z}^{\bar{\Delta}_j+\|\vec{l}\|-\bar{\Delta}_1-\bar{\Delta}_2} L_k L_{-k_1}\cdots L_{-k_r}\bar{L}_{-l_1}\cdots\bar{L}_{-l_s}|\Phi_j\rangle$$
$$= \sum_J C_{21}^J (k\Delta_2-\Delta_1+\Delta_j+\|\vec{k}\|) z^{\Delta_j+\|\vec{k}\|+k-\Delta_1-\Delta_2}\bar{z}^{\bar{\Delta}_j+\|\vec{l}\|-\bar{\Delta}_1-\bar{\Delta}_2} L_{-k_1}\cdots L_{-k_r}\bar{L}_{-l_1}\cdots\bar{L}_{-l_s}|\Phi_j\rangle.$$

We see that these equations do not involve the $\bar{z}$ variable and cannot fix the overall normalization. Let

$$\begin{aligned}|\beta_{j21},K\rangle &= \sum_{\vec{k},\ \|\vec{k}\|=K} \beta_{j21}^{\vec{k}} L_{-k_1}\cdots L_{-k_r}|\Delta_j\rangle,\\ |\bar{\beta}_{j21},K\rangle &= \sum_{\vec{k},\ \|\vec{k}\|=K} \bar{\beta}_{j21}^{\vec{k}} \bar{L}_{-k_1}\cdots \bar{L}_{-k_r}|\bar{\Delta}_j\rangle\end{aligned} \tag{21}$$

Then we have

$$\begin{aligned}L_k|\beta_{j21},K\rangle &= (k\Delta_2-\Delta_1+\Delta_j+K-k)|\beta_{j21},K-k\rangle,\\ \bar{L}_k|\bar{\beta}_{j21},K\rangle &= (k\bar{\Delta}_2-\bar{\Delta}_1+\bar{\Delta}_j+K-k)|\bar{\beta}_{j21},K-k\rangle.\end{aligned} \tag{22}$$

In fact, it is sufficient to only impose these equations for $k=1,2$. All other equations follow from these two due to the commutation relations. The number of equations on each level is greater than the number of variables so that the system is overdetermined. It can be shown that for generic values of Δ_j the coefficients are uniquely determined by the system. For degenerate values $\Delta_j=\Delta_{mn}$ for $K\ge mn$ the solution exists only if the fusion rule holds, and the vectors $|\beta_{j21},K\rangle$ are defined modulo the corresponding submodule.

Define the function

$$F\left(\begin{matrix}\Delta_1 & \Delta_4\\ \Delta_2 & \Delta_3\end{matrix}\quad \Delta_j\middle| z\right) = \sum_{\vec{k}} \gamma_{43j}^{\vec{k}}\beta_{j21}^{\vec{k}} z^{\Delta_j+\|\vec{k}\|-\Delta_1-\Delta_2}, \tag{23}$$

which is called the (4-point) *conformal block*. I wand to stress that the conformal block is uniquely defined in the representation theory of the Virasoro algebra, and does not involve any details of the field theory. It means that it is a well-defined mathematical object. In fact, there exist several approaches for calculating it, such as free field representation (for some special values of dimensions), Al. Zamolodchikov's recurrence relations and, the most recent, the AGT construction.

The four-point correlation function is expressed in terms of the conformal blocks as follows:

$$\langle\Phi_4(\infty)\Phi_3(1)\Phi_2(z,\bar{z})\Phi_1(0)\rangle = \sum_j C_{43j}C_{21}^j F\left(\begin{matrix}\Delta_1 & \Delta_4\\ \Delta_2 & \Delta_3\end{matrix}\quad \Delta_j\middle| z\right) F\left(\begin{matrix}\bar{\Delta}_1 & \bar{\Delta}_4\\ \bar{\Delta}_2 & \bar{\Delta}_3\end{matrix}\quad \bar{\Delta}_j\middle| \bar{z}\right). \tag{24}$$

Analogously,

$$\langle\Phi_4(\infty)\Phi_1(0)\Phi_2(z,\bar{z})\Phi_3(1)\rangle = \sum_j C_{41j}C_{23}^j F\left(\begin{matrix}\Delta_1 & \Delta_4\\ \Delta_2 & \Delta_3\end{matrix}\quad \Delta_j\middle| 1-z\right) F\left(\begin{matrix}\bar{\Delta}_1 & \bar{\Delta}_4\\ \bar{\Delta}_2 & \bar{\Delta}_3\end{matrix}\quad \bar{\Delta}_j\middle| 1-\bar{z}\right). \tag{25}$$

Comparing them, we obtain the crossing symmetry equation [1][2]

$$\sum_j C_{43j}C^j_{21}F\left(\begin{matrix}\Delta_1 & \Delta_4 \\ \Delta_2 & \Delta_3\end{matrix}\quad\Delta_j\middle|\,z\right)F\left(\begin{matrix}\bar\Delta_1 & \bar\Delta_4 \\ \bar\Delta_2 & \bar\Delta_3\end{matrix}\quad\bar\Delta_j\middle|\,\bar z\right)$$
$$=\sum_j C_{14j}C^j_{23}F\left(\begin{matrix}\Delta_3 & \Delta_4 \\ \Delta_2 & \Delta_1\end{matrix}\quad\Delta_j\middle|\,1-z\right)F\left(\begin{matrix}\bar\Delta_3 & \bar\Delta_4 \\ \bar\Delta_2 & \bar\Delta_1\end{matrix}\quad\bar\Delta_j\middle|\,1-\bar z\right). \tag{26}$$

(We used the fact that $\sum s_i = 0$ and (16).) This is an equation for the structure constants C^i_{jk}. Generally, it has multiple solutions, but in the case of minimal and rational minimal models with spinless fields the solution turns out to be unique. The structure constants in this case were calculated by Vl. Dotsenko and V. Fateev [2–4]. There solution was based on the fact that the correlation functions of minimal conformal models can be represented in terms of the free field theory:

$$\left\langle\prod_{i=1}^N \Phi_{m_in_i}(x_i)\right\rangle = \text{const}\times$$
$$\times\left\langle :\mathrm{e}^{-(b+b^{-1})\varphi(\infty)}\colon :\mathrm{e}^{\alpha_{-m_N,-n_N}\varphi(x_N)}\colon \prod_{i=1}^{N-1} :\mathrm{e}^{\alpha_{m_in_i}\varphi(x_i)}\colon \left(\int d^2u\ :\mathrm{e}^{b^{-1}\varphi(u)}\colon\right)^r\left(\int d^2v\ :\mathrm{e}^{b\varphi(v)}\colon\right)^s\right\rangle, \tag{27}$$

where

$$r=\frac12\left(\sum_{i=1}^{N-1}m_i-m_N\right),\qquad s=\frac12\left(\sum_{i=1}^{N-1}n_i-n_N\right).$$

The surface integrals can be calculated by reducing to products of pairs of contour integrals in complex variables, and this decomposition turns out to be just decomposition of the form (24). It is very interesting subject, but I have to omit it due to the lack of time.

Nevertheless, I want to explain one important point. Let $\Delta_2=\Delta_{mn}$, while other dimension will be left arbitrary. Then the correlation function (24) satisfies a linear differential equation of the order mn. Since this equation is analytic in z, the conformal block $F\left(\begin{matrix}\Delta_1 & \Delta_4 \\ \Delta_{mn} & \Delta_3\end{matrix}\quad\Delta_j\middle|\,z\right)$ satisfies the same equation. This equation possesses just mn independent solution. On the other hand according to the fusion rule we know that

$$\alpha_j=\alpha_1+b^{-1}k+bl,\qquad k=-m,-m+2,\dots,m,\qquad l=-n,-n+2,\dots,n.$$

We have mn independent conformal blocks with different monodromy properties. Indeed, if we take z and move them around 0 we get

$$F\left(\begin{matrix}\Delta_1 & \Delta_4 \\ \Delta_{mn} & \Delta_3\end{matrix}\quad\Delta_j\middle|\,\mathrm{e}^{2\pi\mathrm{i}}z\right)=\mathrm{e}^{2\pi\mathrm{i}(\Delta_j-\Delta_{mn}-\Delta_1)}F\left(\begin{matrix}\Delta_1 & \Delta_4 \\ \Delta_{mn} & \Delta_3\end{matrix}\quad\Delta_j\middle|\,z\right). \tag{28}$$

These conformal blocks form a basis in the space of solution to the differential equation. The expression (24) guarantees us that the correlation functions is invariant under this monodromy, if $\bar z = z^*$. But to be sure that it is unique-valued we have to guarantee that the monodromy invariant if we mode the variable z around 1. This is not clear, because the monodromy properties of conformal blocks may be given by a non-diagonal matrix about $z=1$. And at this point the crossing symmetry plays role. Indeed, the r.h.s. of the crossing equation (26) is invariant under such monodromy. The conformal blocks $F\left(\begin{matrix}\Delta_3 & \Delta_4 \\ \Delta_{mn} & \Delta_1\end{matrix}\quad\Delta_j\middle|\,1-z\right)$ form another basis in the space of solution to the differential equation with diagonal monodromy about $z=1$. The two basis are related by a linear transformation:

$$F\left(\begin{matrix}\Delta_1 & \Delta_4 \\ \Delta_{mn} & \Delta_3\end{matrix}\quad\Delta_j\middle|\,z\right)=X\left(\begin{matrix}\Delta_1 & \Delta_4 \\ \Delta_{mn} & \Delta_3\end{matrix}\middle|\begin{matrix}\Delta_{j'}\\ \Delta_j\end{matrix}\right)F\left(\begin{matrix}\Delta_3 & \Delta_4 \\ \Delta_{mn} & \Delta_1\end{matrix}\quad\Delta_{j'}\middle|\,1-z\right). \tag{29}$$

[2]In the derivation here we used slightly different expansion: we expanded the four-point functions about $x_1=(0,0)$ in the l.h.s. and about $x_3=(1,1)$ rather than about $x_2=(z,\bar z)$ in both side, and used the translation invariance. We have done it to get rid of numerous extra minus signs in the intermediate formulas. You can easily check that it is completely equivalent to (2) in the conformal case.

The structure constants then satisfy a system of bilinear algebraic equations

$$\sum_j C_{43j} C_{21}^{j} X \left(\begin{array}{cc|c} \Delta_1 & \Delta_4 & \Delta_{j'} \\ \Delta_{mn} & \Delta_3 & \Delta_j \end{array} \right) X \left(\begin{array}{cc|c} \Delta_1 & \Delta_4 & \Delta_{j''} \\ \Delta_{mn} & \Delta_3 & \Delta_j \end{array} \right) = C_{14j'} C_{23}^{j'} \delta_{j'j''}. \tag{30}$$

Together with the normalization, e.g. $C_{ij}^0 = g_{ij} = \delta_{ij}$ it gives the structure constants, if we know the X constants. Just the X constants can be fixed by means of the free field representation.

It is conjectured that the linear transformation like (29) holds for arbitrary values of Δ_2, where there is no differential equation. In the case of the Liouville theory $c > 25$ it was conjectured to be an integral transformation. Nevertheless, no method other than the free field representation makes it possible to compute the X coefficients precisely. But the free field representation only works for special values of Δ_i since the numbers r and s must be integer. Thus it is an open question.

References

[1] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, *Nucl. Phys.* **B241**, 333 (1984).

[2] Vl. S. Dotsenko and V. A. Fateev, *Nucl. Phys.* **B240**, 312 (1984).

[3] Vl. S. Dotsenko and V. A. Fateev, *Nucl. Phys.* **B251**, 691 (1985).

[4] Vl. S. Dotsenko and V. A. Fateev, *Phys. Lett.* **B154**, 291 (1985).

Problems

1. Find the coefficients β_{j21}^1, β_{j21}^2 and $\beta_{j21}^{1,1}$ by solving equations (22) explicitly. Show that for $\Delta_j = \Delta_{11}, \Delta_{21}, \Delta_{12}$ the corresponding fusion rules follow from the equations.