Lecture 6
Conformal field theory: fusion rules, conformal blocks and crossing symmetry

Let us extract some information from the level 2 differential equation

$$\left(\frac{\partial^2}{\partial z^2}+\lambda\sum_{i=1}^N\left(\frac{\Delta_i}{(z_i-z)^2}-\frac{1}{z_i-z}\frac{\partial}{\partial z_i}\right)\right)\langle\Phi_1(z_1,\bar z_1)\ldots\Phi_N(z_N,\bar z_N)\Phi(z,\bar z)\rangle=0$$

$$\text{where } \lambda=b^{\mp 2} \text{ for } \Delta_\Phi=\Delta_{21},\Delta_{12}. \quad (1)$$

Consider the limit $z\to z_i$ and keep the leading singular terms only:

$$\left(\frac{\partial^2}{\partial z^2}+\frac{\lambda\Delta_i}{(z-z_i)^2}-\frac{\lambda}{z-z_i}\frac{\partial}{\partial z}\right)\Phi(z)\Phi_i(z_i)=0. \quad (2)$$

For a while we forget about the left variables $\bar z,\bar z_i$. Later I will explain how to restore them.

The equation

$$f''(z)-\lambda z^{-1}f'(z)+\lambda\Delta z^{-2}f(z)=0$$

is homogeneous in z, so that its solution is $f(z)=A_1z^{\gamma_1}+A_2z^{\gamma_2}$ with arbitrary A_i and some particular γ_i. After substituting z^γ we obtain $\gamma^2-(\lambda+1)\gamma+\lambda\Delta=0$ so that

$$\gamma_{1,2}=\frac{\lambda+1\pm\sqrt{\lambda^2-2(2\Delta-1)\lambda+1}}{2}.$$

If we assume the operator product of the form

$$\Phi(z')\Phi_i(z_i)=C^{(1)}(z'-z)^{\Delta_i^{(1)}-\Delta_\Phi-\Delta_1}\Phi_i^{(1)}(z_i)+C^{(2)}(z'-z)^{\Delta_i^{(2)}-\Delta_\Phi-\Delta_1}\Phi_i^{(2)}(z_i)+\cdots. \quad (3)$$

We obtain

$$\Delta_i^{(1,2)}=\Delta_i+\Delta+\gamma_{1,2}. \quad (4)$$

This equation looks cumbersome, but it becomes trivial in the parametrization $\Delta=\alpha(b+b^{-1}-\alpha)$:

$$\alpha_i^{(1,2)}=\begin{cases}\alpha_i\pm\alpha_{21}, & \text{if } \Delta_\Phi=\Delta_{21};\\ \alpha_i\pm\alpha_{12}, & \text{if } \Delta_\Phi=\Delta_{12}.\end{cases} \quad (5)$$

Let us parametrize the primary field with the parameter α: $\Phi_\alpha=\Phi_{b+b^{-1}-\alpha}=\Phi_\Delta$. Then

$$\begin{aligned}[\Phi_\alpha][\Phi_{21}]&=[\Phi_{\alpha+\alpha_{21}}]+[\Phi_{\alpha-\alpha_{21}}],\\ [\Phi_\alpha][\Phi_{12}]&=[\Phi_{\alpha+\alpha_{12}}]+[\Phi_{\alpha-\alpha_{12}}].\end{aligned} \quad (6)$$

Here square bracket means the *conformal families*, i.e. sets of all descendants of the corresponding primary operators.

Now let us try to answer the question: what happens, when $\alpha=\alpha_{mn}$ in (6)? Formally, we obtain

$$\begin{aligned}[\Phi_{mn}][\Phi_{21}]&=[\Phi_{m+1,n}]+[\Phi_{m-1,n}],\\ [\Phi_{mn}][\Phi_{12}]&=[\Phi_{m,n+1}]+[\Phi_{m,n-1}].\end{aligned} \quad (7)$$

But there is a subtlety: consider the product $[\Phi_{12}][\Phi_{21}]$. According to the first line of (7) we obtain $[\Phi_{22}]+[\Phi_{02}]$. But if we apply the second rule we get $[\Phi_{22}]+[\Phi_{20}]$. Consistency demands that $[\Phi_{12}][\Phi_{21}]=[\Phi_{22}]$. Indeed, the families $[\Phi_{02}]$ and $[\Phi_{20}]$ are nondegenerate, and it would be strange, if they appeared in fusion rules of degenerate families. We see that the rule (7) should be improved.

To do it return to general values of α. Consider the product $[\Phi_\alpha][\Phi_{21}][\Phi_{21}]$. On one hand it is

$$[\Phi_\alpha][\Phi_{21}][\Phi_{21}]=[\Phi_{\alpha+\alpha_{21}}][\Phi_{21}]+[\Phi_{\alpha-\alpha_{21}}][\Phi_{21}]=[\Phi_{\alpha+2\alpha_{21}}]+[\Phi_\alpha]+[\Phi_{\alpha-2\alpha_{21}}].$$

One the other hand by fusing two last factors we obtain

$$[\Phi_\alpha][\Phi_{21}][\Phi_{21}]=[\Phi_\alpha][\Phi_{11}]+[\Phi_\alpha][\Phi_{31}].$$

It is evident that $[\Phi_\alpha][\Phi_{11}]=[\Phi_\alpha]$. We conclude that

$$[\Phi_\alpha][\Phi_{31}]=[\Phi_{\alpha+2\alpha_{21}}]+[\Phi_\alpha]+[\Phi_{\alpha-2\alpha_{21}}]. \tag{8}$$

By continuing in tha same manner we obtain

$$[\Phi_\alpha][\Phi_{mn}]=\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}[\Phi_{\alpha+(m-1-2i)\alpha_{21}+(m-1-2j)\alpha_{12}}]. \tag{9}$$

Now specify α to $\alpha_{m'n'}$. Then the last equation reads

$$[\Phi_{m'n'}][\Phi_{mn}]=\sum_{i=0}^{m-1}\sum_{j=0}^{n-1}[\Phi_{m'+m-1-2i,n'+n-1-2j}]=\sum_{\substack{m''=m'-m+1\\ (\mathrm{mod}\ 2)}}^{m+m'-1}\ \sum_{\substack{n''=n'-n+1\\ (\mathrm{mod}\ 2)}}^{n+n'-1}[\Phi_{m''n''}].$$

Well, but we may interchange the pairs m,n and m',n'. It imposes another restriction to the terms that may appear in the r.h.s. It results

$$[\Phi_{m'n'}][\Phi_{mn}]=\sum_{\substack{m''=|m-m'|+1\\ (\mathrm{mod}\ 2)}}^{m+m'-1}\ \sum_{\substack{n''=|n-n'|+1\\ (\mathrm{mod}\ 2)}}^{n+n'-1}[\Phi_{m''n''}]. \tag{10}$$

The system of operators Φ_{mn} with $m,n>0$ is closed with respect to the fusion rule (10). It makes it possible to define the so called *minimal conformal models*, which contain just one primary field Φ_{mn} of each conformal dimension $\Delta=\bar{\Delta}=\Delta_{mn}$.[1] Nevertheless, generic minimal conformal models possess fields with negative conformal dimensions. This means that norms of some vectors are negative. Indeed,

$$\|L_{-1}|\Delta\rangle\|^2=\langle\Delta|L_1L_{-1}|\Delta\rangle=2\langle\Delta|L_0|\Delta\rangle=2\Delta$$

is negative for $\Delta<0$. Though nonunitary models have some applications, construction of unitary ones is our priority.

Recall the definition of α_{mn}:

$$\alpha_{mn}=\frac{1-m}{2}b^{-1}+\frac{1-n}{2}b. \tag{11}$$

All these numbers are different for $m,n>0$, if b^2 is positive or negative and irrational. But if

$$b^2=-\frac{r}{s}\quad \text{or}\quad c=1-\frac{6(s-r)^2}{rs}, \tag{12}$$

with r,s coprime positive integers (for definiteness we will assume $r<s$), we have

$$\alpha_{m+r,n+s}=\alpha_{mn}. \tag{13}$$

For conformal dimensions

$$\Delta_{mn}=\frac{(sm-rm)^2-(s-r)^2}{4rs} \tag{14}$$

we have

$$\begin{aligned}\Delta_{mn}&=\Delta_{m+r,n+s}\\ \Delta_{mn}&=\Delta_{r-m,s-n}.\end{aligned} \tag{15}$$

Due to the last reflection equation the fusion rules are closed on the set of families

$$[\Phi_{mn}],\quad 1\le m\le r-1,\quad 1\le n\le s-1. \tag{16}$$

[1] There are also theories with non-diagonal primary fields ($\Delta\neq\bar{\Delta}$), but we will not discuss them here except one example.

The fusion rule consistent with the reflection property is

$$[\Phi_{m'n'}][\Phi_{mn}] = \sum_{\substack{m''=\max(|m-m'|+1,m+m'-r+1)\\ (\mathrm{mod}\ 2)}}^{\min(m+m'-1,r-|m-m'|-1)} \sum_{\substack{n''=\max(|n-n'|+1,n+n'-s+1)\\ (\mathrm{mod}\ 2)}}^{\min(n+n'-1,s-|n-n'|-1)} [\Phi_{m''n''}]. \tag{17}$$

The *rational minimal models* are obtained after identification

$$\Phi_{r-m,s-n}(z,\bar z) = \Phi_{mn}(z,\bar z) \tag{18}$$

and bounding the set of primary operators to the *Kac table* (16). Note that operators off the Kac table, which can be obtained from operators within the Kac table by any composition of the reflections $m \to -m$, $m \to 2r-m$, $n \to -n$, $n \to 2s-n$ has the same dimensions as appropriate null vectors in the Kac table:

$$\Delta_{-m,n} = \Delta_{m,-n} = \Delta_{mn} + mn, \qquad \Delta_{2r-m,n} = \Delta_{m,2s-n} = \Delta_{mn} + (r-m)(s-n).$$

In fact, every Verma module in the Kac table contains an infinite number of null vector. The first two are on the levels mn and $(r-m)(s-n)$. But, as we see, they again generate degenerate Verma modules. These two modules intersect and have two null vectors in their intersection, which possess the same property.

Special role is played by the 'self-reflected' operators (e.g. Φ_{0n}), which appear in the so called *logarithmic* conformal field theory.

Theorem. *All irreducible representations in the Kac table are unitary, if and only if* $s = r + 1$.[2]

We will not prove the theorem. We only make sure that for $s = r+1$ all dimensions are positive. Indeed, $\Delta_{mn} < 0$ if and only if

$$|sm - rn| < s - r.$$

If $s - r = 1$ the only value of $sm - rn$ that satisfies this condition is zero. But $sm - rn \neq 0$ since s and r are mutually simple. For $s - r > 1$ this argument fails, since $sm - rn$ can be nonzero.

Now we consider some examples.

The simplest one is $r = 2, s = 3$, which corresponds to $c = 0$. Its Kac table looks like

$$\boxed{\begin{array}{c} 0 \\ \hline 0 \end{array}} = \boxed{\begin{array}{c} I \\ \hline I \end{array}}$$

In the left table we wrote conformal dimensions, while in the write one the corresponding operators. We have the only primary field $\Phi_{11} = 1 \equiv I$, and the corresponding Verma module contains two lowest null vectors $L_{-1}|0\rangle$ and $(L_{-2} - \frac{3}{2}L_{-1}^2)|0\rangle$. Since all L_{-k} are obtained as commutators of L_{-1} and L_{-2}, these two null vectors generate all vectors in the module except the vacuum one. Hence, the irreducible representation is one-dimensional and the theory is void: its space of operators consists of the unit operator and its space of states only consists of vacuum.[3]

Now consider the case $r = 3, s = 4$, which corresponds to $c = 1/2$. It is just the central charge of the free fermion. The Kac table reads

$$\begin{array}{|c|c|} \hline \frac{1}{2} & 0 \\ \hline \frac{1}{16} & \frac{1}{16} \\ \hline 0 & \frac{1}{2} \\ \hline \end{array} = \begin{array}{|c|c|} \hline \varepsilon & \mathbf{1} \\ \hline \sigma & \sigma \\ \hline \mathbf{1} & \varepsilon \\ \hline \end{array} = \begin{array}{|c|c|} \hline \varepsilon & I \\ \hline \mu & \mu \\ \hline I & \varepsilon \\ \hline \end{array},$$

which gives the dimensions of the operators $1, \varepsilon, \sigma$ or $1, \varepsilon, \mu$. From the point of view of bootstrap conformal theory there is no difference between these to sets. They form two same operator algebras. The set of mutually local operators $1, \Psi, \bar\Psi, \varepsilon$ (and their descendants) corresponds to a non-diagonal pairing of two chiral parts of the same fields.

[2]We assumed that $s > r$.

[3]Note that it is true just for the minimal model. It admits some extensions. In particular, its extension by logarithmic operators finds an application to the percolation theory (Cardy, 1991).

The next unitary minimal conformal model $r = 4, s = 5$ and $c = \frac{7}{10}$ corresponds to the so called tricritical Ising model:

$$H = -J \sum_{<i,j>} \sigma_i \sigma_j - \mu \sum_i |\sigma_i|, \qquad \sigma_i = 0, \pm 1.$$

Its Kac table reads

$$\begin{array}{|c|c|c|}\hline \frac{3}{2} & \frac{7}{16} & 0 \\ \hline \frac{3}{5} & \frac{3}{80} & \frac{1}{10} \\ \hline \frac{1}{10} & \frac{3}{80} & \frac{3}{5} \\ \hline 0 & \frac{7}{16} & \frac{3}{2} \\ \hline \end{array} = \begin{array}{|c|c|c|}\hline \varepsilon'' & \sigma' & I \\ \hline v = \varepsilon' & \sigma & \varepsilon \\ \hline \varepsilon & \sigma & v = \varepsilon' \\ \hline I & \sigma' & \varepsilon'' \\ \hline \end{array}$$

The operator $v = \varepsilon'$ is the vacancy ($\sigma_i = 0$) density operator or subleading energy density operator. The operator ε' is the sub-subleading energy density operator. The operator σ' is the subleading magnetization operator.

Our last example is the non-unitary $r = 2, s = 5$ model. Its central charge is $c = -22/5$. It is the so called Lee–Yang critical point in the Ising model. Its Kac table is

$$\begin{array}{|c|}\hline 0 \\ \hline -\frac{1}{5} \\ \hline -\frac{1}{5} \\ \hline 0 \\ \hline \end{array} = \begin{array}{|c|}\hline I \\ \hline \Phi \\ \hline \Phi \\ \hline I \\ \hline \end{array}$$

Except the unit operator, it has only one primary operator $\Phi = \Phi_{12} = \Phi_{13}$ of negative dimension.

Problems

1. Assuming that any Verma module with the weight Δ_{mn} in the Kac table contains two null vectors with dimensions $Delta_{mn} + mn$ and $\Delta_{mn} + (r - m)(s - m)$, prove that it contains an infinite set of null vectors of dimensions

$$\begin{aligned} \Delta_{m,n+2sk} &= \Delta_{mn} + (rk - m)(sk + n) + mn \qquad && (k \in \mathbb{Z} \setminus \{0\}), \\ \Delta_{m,-n+2sk} &= \Delta_{mn} + (rk - m)(sk - n) && (k \in \mathbb{Z}). \end{aligned}$$

2. Prove that for the Möbius invariant field theory for any field $\Phi(x)$ of definite dimensions $(\Delta, \bar{\Delta})$

$$\langle \Phi(x) \rangle = 0, \quad \text{if } \Delta \neq 0 \text{ or } \bar{\Delta} \neq 0,$$

and for any two Möbius invariant fields $\Phi_1(x)$ and $\Phi_2(x)$ of dimensions $(\Delta_1, \bar{\Delta}_1)$ and $(\Delta_2, \bar{\Delta}_2)$ respectively

$$\langle \Phi_1(x') \Phi_2(x) \rangle = 0, \quad \text{if } \Delta_1 \neq \Delta_2 \text{ or } \bar{\Delta}_1 \neq \bar{\Delta}_2.$$

The Möbius invariance of the fields means invariance with respect to the transformation

$$\Phi(z, \bar{z}) \to (f'(z))^{\Delta} (\bar{f}'(\bar{z}))^{\bar{\Delta}} \Phi(f(z), (\bar{\mathbf{z}}))$$

with Möbius functions $f, \bar{f}$. In particular, any primary field is Möbius invariant.

3. Prove that for three Möbius invariant fields $\Phi_i(x)$ $(i = 1, 2, 3)$ the following relation holds

$$\begin{aligned} \langle \Phi_3(x_3) \Phi_2(x_2) \Phi_1(x_1) \rangle = C_{123} &(z_2 - z_1)^{\Delta_3 - \Delta_1 - \Delta_2} (z_3 - z_1)^{\Delta_2 - \Delta_1 - \Delta_3} (z_3 - z_2)^{\Delta_1 - \Delta_2 - \Delta_3} \\ &\times (\bar{z}_2 - \bar{z}_1)^{\bar{\Delta}_3 - \bar{\Delta}_1 - \bar{\Delta}_2} (\bar{z}_3 - \bar{z}_1)^{\bar{\Delta}_2 - \bar{\Delta}_1 - \bar{\Delta}_3} (\bar{z}_3 - \bar{z}_2)^{\bar{\Delta}_1 - \bar{\Delta}_2 - \bar{\Delta}_3}. \end{aligned}$$