Lecture 4
Free massless fermion on the plane: operator product expansions

Here we discuss th operator algebra of the fields $\Psi(z), \bar\Psi(\bar z), \sigma(x), \mu(x)$. First, consider the product $\Psi(z')\bar\Psi(\bar z)$. Since the operators β_k anticommute with the operators $\bar\beta_l$, this product is regular,

$$\Psi(z')\bar\Psi(\bar z) = \Psi(z)\bar\Psi(\bar z) + O(z'-z) = \mathrm{i}\varepsilon(x) + O(z'-z), \tag{1}$$

and define the so called *energy operator*. Its name comes from the Ising model, since the perturbation of the model by $\tau\int d^2x\,\varepsilon(x)$ corresponds to the temperature deviation from the critical point: $\tau\sim T-T_c$. In the field theory it is the mass perturbation: $\tau\sim m$.

The product $\Psi(z')\Psi(z)$ can be calculated directly:

$$\Psi(z')\Psi(z) = \frac{1}{z'-z} - (z'-z)\,{:}\Psi\partial\Psi{:} + O\big((z'-z)^2\big). \tag{2}$$

The regular part consists of terms of the form $C_{kl}(z'-z)^{k+l}\,{:}\partial^k\Psi\partial^l\Psi{:}$. The first of these regular terms is written explicitly, since it plays an important role below. Besides, we immediately get

$$\varepsilon(x')\varepsilon(x) = \frac{1}{(z'-z)(\bar z'-\bar z)} + O(1). \tag{3}$$

Now consider the product $\Psi(z')\sigma(x)$. To do it, let $x=0$:

$$\Psi(z')\sigma(0)|0\rangle_{\rm NS} = \Psi(z')|0\rangle_{\rm R} = \sum_{k\le 0}\beta_k z'^{-1/2-k}|0\rangle_{\rm R} = \frac{\mathrm{i}^{-1/2}}{\sqrt2}z'^{-1/2}|1\rangle_{\rm R} + O\big(z'^{1/2}\big),$$

since $b_0|0\rangle_{\rm R} = \frac{1}{\sqrt2}(\mathrm{i}^{1/2}c_0+\mathrm{i}^{-1/2}c_0^+)|0\rangle_{\rm R} = \frac{\mathrm{i}^{-1/2}}{\sqrt2}|1\rangle_{\rm R}$. We obtain

$$\Psi(z')\sigma(x) = \frac{\mathrm{i}^{-1/2}}{\sqrt2}\frac{\mu(x)}{(z'-z)^{1/2}} + O\big((z'-z)^{1/2}\big). \tag{4}$$

Similarly we obtain

$$\Psi(z')\mu(x) = \frac{\mathrm{i}^{1/2}}{\sqrt2}\frac{\sigma(x)}{(z'-z)^{1/2}} + O\big((z'-z)^{1/2}\big). \tag{5}$$

The operator product expansions for $\bar\Psi(\bar z)$ differ by substitution $z\to\bar z$, $\mathrm{i}^{\pm1/2}\to\mathrm{i}^{\mp1/2}$.

An important feature of (4) and (5) is that the contain half-integer powers of $(z'-z)$. It means that if we move x' around x and return it to the initial place in the Euclidean plane, the corresponding correlation function multiplies by -1. Surely, this is the immediate consequence of the Ramond condition on the plane. But it means that the operators $\sigma(x), \mu(x)$ are nonlocal with respect to $\Psi(z), \bar\Psi(\bar z)$. In fact, they can be written in terms of the 'basic' fields, but in a rather complicated form. We will not need it to obtain results at the critical point. It is important that the definition of these operators contain contour integrals, so that the contour plays the role of a cut on which the sign of $\Psi, \bar\Psi$ changes.

Now let us try to obtain the first term of decomposition of products like $\sigma(x')\sigma(x)$. It will contain the unit operator as well as the energy operator $\varepsilon(x)$. To obtain it we use the matrix elements

$${}_{\rm R}\langle 0|\epsilon(x)|0\rangle_{\rm R} = -\frac{1}{2(z\bar z)^{1/2}}, \qquad {}_{\rm R}\langle 1|\epsilon(x)|1\rangle_{\rm R} = \frac{1}{2(z\bar z)^{1/2}}.$$

Simultaneously, we obtain the coefficients in $\varepsilon(x')\sigma(x)$ and $\varepsilon(x')\mu(x)$. We have

$$\sigma(x')\sigma(x) = \frac{1}{(z'-z)^{1/8}(\bar z'-\bar z)^{1/8}} - \frac12(z'-z)^{3/8}(\bar z'-\bar z)^{3/8}\varepsilon(x) + O\big((z'-z)^{7/8}(\bar z'-\bar z)^{7/8}\big), \tag{6}$$

$$\mu(x')\mu(x) = \frac{1}{(z'-z)^{1/8}(\bar z'-\bar z)^{1/8}} + \frac12(z'-z)^{3/8}(\bar z'-\bar z)^{3/8}\varepsilon(x) + O\big((z'-z)^{7/8}(\bar z'-\bar z)^{7/8}\big), \tag{7}$$

$$\epsilon(x')\sigma(x) = -\frac12\frac{\sigma(x)}{(z'-z)^{1/2}(\bar z'-\bar z)^{1/2}} + O\big((z'-z)^{1/2}(\bar z'-\bar z)^{1/2}\big), \tag{8}$$

$$\epsilon(x')\mu(x) = \frac{1}{2}\frac{\mu(x)}{(z'-z)^{1/2}(\bar z'-\bar z)^{1/2}} + O\big((z'-z)^{1/2}(\bar z'-\bar z)^{1/2}\big). \tag{9}$$

The last operator product is

$$\mu(x')\sigma(x) = \frac{\mathrm{i}^{1/2}}{\sqrt{2}}\frac{(z'-z)^{3/8}}{(\bar z'-\bar z)^{1/8}}\Psi(z) + \frac{\mathrm{i}^{-1/2}}{\sqrt{2}}\frac{(\bar z'-\bar z)^{3/8}}{(z'-z)^{1/8}}\bar\Psi(\bar z) + O\big((z'-z)^{7/8}(\bar z'-\bar z)^{7/8}\big). \tag{10}$$

These operators are not mutually local as well.

The set of operator products expansions (1) — (10) is not a complete set of operator product expansion, since they ignore an infinite set of operators, which denoted by $O(\cdots)$. Nevertheless, we will see that, together with the conformal symmetry, they, in a sense, completely describe the theory. Coefficients at the terms in the r.h.s. are called *structure constants* of the theory. They, in fact, define all operator products.

Now summarize the results as follows. The theory of a free fermion contains a set of operators, which are not all mutually local. They contains three subsets of mutually local operators

$$\begin{array}{lllll} [1] & [\Psi] & [\bar\Psi] & [\varepsilon] & \text{(free fermion)} \\ [1] & [\varepsilon] & [\sigma] & & \text{(Ising model)} \\ [1] & [\varepsilon] & [\mu] & & \text{(`dual' Ising model)} \end{array} \tag{11}$$

Here brackets mean the 'families' of operators, i.e. sets of operators, whose conformal dimensions differ by integers. The precise sense of this term will be explained later in this lecture. Note that the comments at the right do not mean that these are three different field theories. On the plane these are three different field-theoretical descriptions of the same quantum mechanics.

The OPEs (1) — (10) provide also the fusion rules between these families. For these three sets of local fields they read:

$$\begin{aligned} &1.\ [\Psi]\times[\Psi]=[1],\ [\bar\Psi]\times[\bar\Psi]=[1],\ [\Psi]\times[\varepsilon]=[\bar\Psi],\ [\bar\Psi]\times[\varepsilon]=[\Psi],\ [\varepsilon]\times[\varepsilon]=[1];\\ &2.\ [\varepsilon]\times[\varepsilon]=[1],\ [\varepsilon]\times[\sigma]=[\sigma],\ [\sigma]\times[\sigma]=[1]+[\varepsilon];\\ &3.\ [\varepsilon]\times[\varepsilon]=[1],\ [\varepsilon]\times[\mu]=[\mu],\ [\mu]\times[\mu]=[1]+[\varepsilon]. \end{aligned} \tag{12}$$

The energy-momentum tensor of the free massless fermion is given by

$$T(z) = -2\pi T_{zz} = -\frac{1}{2}{:}\Psi\,\partial\Psi{:}\,, \qquad \bar T(\bar z) = -2\pi T_{\bar z\bar z} = -\frac{1}{2}{:}\bar\Psi\,\bar\partial\bar\Psi{:}\,. \tag{13}$$

The OPT of $T(z)$ reads

$$T(z')T(z) = \frac{1/4}{(z'-z)^4} + \frac{2T(z)}{(z'-z)^2} + \frac{\partial T(z)}{z'-z} + O(1). \tag{14}$$

It only differs from the boson case by the numerator $1/4$ of the first term, which is twice smaller than that for the boson field.

The corresponding operators L_k are given by

$$L_k = \frac{1}{2}\sum_{l\in\mathbb{Z}+\frac{\delta}{2}}\left(l-\frac{k}{2}\right){:}\beta_{k-l}\beta_l{:}\,, \qquad \bar L_k = \frac{1}{2}\sum_{l\in\mathbb{Z}+\frac{\delta}{2}}\left(l-\frac{k}{2}\right){:}\bar\beta_{k-l}\bar\beta_l{:}\,. \tag{15}$$

They again form the Virasoro algebra

$$[L_k, L_l] = (k-l)L_{k+l} + \frac{c}{12}k(k^2-1)\delta_{k+l,0}, \tag{16}$$

but with the *central charge* c equal to $\frac{1}{2}$ instead of 1 for the free boson.

We want to write operator product expansions of $T(z), \bar T(\bar z)$ with other fields. First, write a general form. A field $\Phi(x)$ is called *primary*, if its OPE with the energy-momentum tensor reads

$$\begin{aligned} T(z')\Phi(z,\bar z) &= \frac{\Delta\Phi(z,\bar z)}{(z'-z)^2} + \frac{\partial\Phi(z,\bar z)}{z'-z} + O(1),\\ \bar T(\bar z')\Phi(z,\bar z) &= \frac{\bar\Delta\Phi(z,\bar z)}{(\bar z'-\bar z)^2} + \frac{\bar\partial\Phi(z,\bar z)}{\bar z'-\bar z} + O(1), \end{aligned} \tag{17}$$

where $\Delta, \bar{\Delta}$ are numbers call *conformal dimensions*. In terms of $L_k, \bar{L}_k$ these condition can be rewritten as

$$\begin{aligned} [L_k, \Phi(z,\bar z)] &= (k+1)\Delta z^k \Phi(z,\bar z) + z^{k+1}\partial\Phi(z,\bar z), \\ [\bar L_k, \Phi(z,\bar z)] &= (k+1)\bar\Delta \bar z^k \Phi(z,\bar z) + \bar z^{k+1}\bar\partial\Phi(z,\bar z). \end{aligned} \tag{18}$$

It is easy to check (solve Problem 3!) that the operators $1, \Psi, \bar\Psi, \varepsilon, \sigma, \mu$ are primary with the conformal dimensions

$$\begin{aligned} 1 &: \Delta_1 = \bar\Delta_1 = 0; \\ \Psi &: \Delta_\Psi = \tfrac12,\ \bar\Delta_\Psi = 0; \\ \bar\Psi &: \Delta_{\bar\Psi} = 0,\ \bar\Delta_{\bar\Psi} = \tfrac12; \\ \sigma, \mu &: \Delta_\sigma = \bar\Delta_\sigma = \Delta_\mu = \bar\Delta_\mu = \tfrac{1}{16}. \end{aligned} \tag{19}$$

Recall that the exponential operators $V_p(x)$ in the free boson theory are also primary:

$$V_p = {:}\mathrm{e}^{\mathrm{i}p\varphi}{:} : \ \Delta_p = \bar\Delta_p = p(p - \mathcal{Q}). \tag{20}$$

We assumed that the energy-momentum tensor has the modified form from Problem 4 of Lecture 2.

Up to now we obtained the OPE (17) and the commutation relation (18) in a heuristic way. But what is their true sense? Two understand it consider the state

$$|\Phi\rangle = \Phi(0)|0\rangle, \tag{21}$$

where $|0\rangle$ is the true vacuum of the theory (e.g. $|p=0\rangle$ for the free boson or $|0\rangle_{\mathrm{NS}}$ for the free fermion). Then from (18) we immediately obtain

$$L_k|\Phi\rangle = \bar L_k|\Phi\rangle = 0 \quad (k>0), \qquad L_0|\Phi\rangle = \Delta|\Phi\rangle, \qquad \bar L_0|\Phi\rangle = \bar\Delta|\Phi\rangle. \tag{22}$$

This is nothing but the definition of the *highest weight vector* in the tensor sum of the two Virasoro algebras. The whole highest weight representation is the span of vectors of the form

$$|\Phi, \vec k, \vec l\rangle = L_{-k_1}\ldots L_{-k_r}\bar L_{-l_1}\ldots \bar L_{-l_s}|\Phi\rangle \quad (k_i, l_i > 0). \tag{23}$$

Each vector (23) is an eigenvector of the operators $L_0, \bar L_0$. Indeed, $[L_0, L_{-k}] = kL_{-k}$, $[\bar L_0, \bar L_{-k}] = k\bar L_{-k}$. Hence,

$$L_0|\Phi,\vec k,\vec l\rangle = \left(\Delta + \sum k_i\right)|\Phi,\vec k,\vec l\rangle, \qquad \bar L_0|\Phi,\vec k,\vec l\rangle = \left(\bar\Delta + \sum l_i\right)|\Phi,\vec k,\vec l\rangle. \tag{24}$$

The pair of integers $(\sum k_i, \sum l_i)$ is called the *level* of the vector.

This construction makes it possible to split the space of states of any conformal field theory into a sum of (not necessarily irreducible) representations of the product of two Virasoro algebras. The highest weight condition (22) simply reflects the physical fact that the spectrum of the Hamiltonian of any reasonable theory on the cylinder must be bounded below.

The state-operator correspondence (21) says that the construction (23) makes also possible to classify the space of operators in terms of the Virasoro algebra representations. Namely, define the operator $(L_{-k_1}\cdots\bar L_{-l_1}\cdots\Phi)(x)$ according to

$$(L_{-k_1}\cdots L_{-k_r}\bar L_{-l_1}\cdots\bar L_{-l_s}\Phi)(0)|0\rangle = |\Phi,\vec k,\vec l\rangle. \tag{25}$$

Then the space of operators split into a sum of highest weight representations of the product of two Virasoro algebras. In the next lecture we will discuss the structure of the representations, while now let us derive equations (17) (and, hence (18)) from the highest weight condition (22). Consider the vector

$$T(z)|\Phi\rangle = \sum_{k\in\mathbb{Z}} L_k z^{-2-k}|\Phi\rangle = \sum_{k=-\infty}^{0} L_k z^{-2-k}|\Phi\rangle = z^{-2}L_0|\Phi\rangle + z^{-1}L_{-1}|\Phi\rangle + O(1).$$

The action of L_0 is determined by (22), but the action of L_{-1} demands some more physical intuition. Note that

$$L_{-1}\Phi(0) = \oint \frac{dz}{2\pi\mathrm{i}} T(z)\Phi(0) = \mathrm{i}\oint dz\, T_{zz}\Phi(0)$$
$$= \mathrm{i}\int_{-\infty}^{\infty} dx^1\, \Phi(0)T_{zz}(x^1 - \mathrm{i}0) - \mathrm{i}\int_{-\infty}^{\infty} dx^1\, T_{zz}(x^1 + \mathrm{i}0)\Phi(0) = -\mathrm{i}[P_z, \Phi(0)] = \partial\Phi(0).$$

Hence,

$$T(z)|\Phi\rangle = z^{-2}\Delta|\Phi\rangle + z^{-1}|\partial\Phi\rangle + O(1),$$

which corresponds to the first line of (17).

Problems

1. Derive the expansion (2). Show that it is the same in both the NS and R sector.
2. Derive (14) and (15).
3. Prove that the operators $1, \Psi, \bar{\Psi}, \varepsilon, \sigma, \mu$ are primary with the conformal dimensions given by (19).