**Lecture 2**
**Correlation functions and operator product expansions**

In the last lecture we found that
$$|p\rangle = \mathrm{e}^{\mathrm{i}q_0 p}|0\rangle. \tag{1}$$
We want to use this fact to introduce a new class of operators. For this purpose rewrite (1) as follows
$$|p\rangle = \mathrm{e}^{\mathrm{i}q_0 p}|0\rangle = \ :\mathrm{e}^{\mathrm{i}p\varphi(\xi)}:|0\rangle\Big|_{\xi^0\to-\infty} = \ :\mathrm{e}^{\mathrm{i}p\varphi(x=0)}:|0\rangle.$$
Indeed, the contribution of $\alpha_k$ $(k>0)$ vanishes since they kill the vacuum, while the contribution of $\alpha_{-k}$ vanishes since it contains the factor $z^k\to 0$.

The exponential operators are very important and deserve a special notation:
$$V_p(x) = \ :\mathrm{e}^{\mathrm{i}p\varphi(x)}: \ = \mathrm{e}^{\mathrm{i}pQ}(z\bar z)^{-2\mathrm{i}pP}\exp\sum_{k>0}\frac{-p(\alpha_{-k}z^k+\bar\alpha_{-k}\bar z^k)}{k}\exp\sum_{k>0}\frac{p(\alpha_k z^{-k}+\bar\alpha_k\bar z^{-k})}{k}. \tag{2}$$

Consider the product
$$\begin{aligned}V_{p_1}(x')V_{p_2}(x) = {}& \mathrm{e}^{\mathrm{i}p_1Q}\boxed{(z'\bar z')^{-2\mathrm{i}p_1P}}\exp\sum_{k>0}\frac{-p_1(\alpha_{-k}z'^k+\bar\alpha_{-k}\bar z'^k)}{k}\boxed{\exp\sum_{k>0}\frac{p_1(\alpha_k z'^{-k}+\bar\alpha_k\bar z'^{-k})}{k}}\\ &\times\boxed{\mathrm{e}^{\mathrm{i}p_2Q}}(z\bar z)^{-2\mathrm{i}p_2P}\boxed{\exp\sum_{k>0}\frac{-p_2(\alpha_{-k}z^k+\bar\alpha_{-k}\bar z^k)}{k}}\exp\sum_{k>0}\frac{p_2(\alpha_k z^{-k}+\bar\alpha_k\bar z^{-k})}{k}.\end{aligned}$$

To render this to the normal ordered form the boxed parts must be commuted (red with red, blue with blue). By using the standard rule
$$\mathrm{e}^f\mathrm{e}^g = \mathrm{e}^{[f,g]}\mathrm{e}^g\mathrm{e}^f,$$
if the commutator $[f,g]$ is a $c$-number, we obtain
$$V_{p_1}(x')V_{p_2}(x) = (z'-z)^{2p_1p_2}(\bar z'-\bar z)^{2p_1p_2}\ :\mathrm{e}^{\mathrm{i}p_1\varphi(x')+\mathrm{i}p_2\varphi(x)}:. \tag{3}$$

More generally,
$$V_{p_N}(x_N):\mathrm{e}^{\mathrm{i}\sum_{i=1}^{N-1}p_i\varphi(x_i)}: \ = \prod_{i=1}^{N-1}(z_N-z_i)^{2p_Np_i}(\bar z_N-\bar z_i)^{2p_Np_i}\times :\mathrm{e}^{\mathrm{i}\sum_{i=1}^{N}p_i\varphi(x_i)}:. \tag{4}$$

Hence,
$$\left\langle\prod_{i=1}^{\overset{\curvearrowleft}{N}}V_{p_i}(x_i)\right\rangle = \prod_{i>j}^{N}(z_i-z_j)^{2p_ip_j}(\bar z_i-\bar z_j)^{2p_ip_j}\times\langle\mathrm{e}^{\mathrm{i}Q\sum_{i=1}^N p_i}\rangle.$$

We have nearly reached our destination, but we have to consider the last factor accurately. Let $p=\sum p_i$ and write
$$\langle\mathrm{e}^{\mathrm{i}pQ}\rangle = \langle 0|\mathrm{e}^{\mathrm{i}pQ}|0\rangle = \langle 0|p\rangle.$$
The r.h.s. must be zero, if $p\ne 0$, and a nonzero constant (e.g. 1), if $p=0$. It is mathematically consistent, but the answer may look strange for a physicist. To advance in a more physical way, let us apply the rule
$$\langle\mathrm{e}^f\rangle = \mathrm{e}^{\frac12\langle f^2\rangle}, \tag{5}$$
if $f$ is linear in the basic oscillator operators. We have
$$\langle\mathrm{e}^{\mathrm{i}pQ}\rangle = \mathrm{e}^{-\frac{p^2}{2}\langle Q^2\rangle} = R^{-2p^2} \tag{6}$$

according to the assumption of the last lecture. Therefore

$$\left\langle \prod_{i=1}^{\widehat{N}} V_{p_i}(x_i) \right\rangle = R^{-2\left(\sum_{i=1}^N p_i\right)^2} \prod_{i>j}^{N} (z_i - z_j)^{2p_ip_j} (\bar{z}_i - \bar{z}_j)^{2p_ip_j}. \tag{7}$$

Since physically $R$ is the scale of the scrap on which the theory lives, the infinite plane corresponds $R \to \infty$. In this limit we reproduce the mathematical answer

$$\left\langle \prod_{i=1}^{\widehat{N}} V_{p_i}(x_i) \right\rangle = \prod_{i>j}^{N} (z_i - z_j)^{2p_ip_j} (\bar{z}_i - \bar{z}_j)^{2p_ip_j} \times \begin{cases} 1, & \sum_{i=1}^N p_i = 0; \\ 0, & \sum_{i=1}^N \neq 0. \end{cases} \tag{8}$$

Let us reproduce this answer in yet more physical (and simple) way. Calculate $\langle \exp \mathrm{i} \sum p_i \varphi(x_i) \rangle$ by means of the functional integral. In fact, the only thing we need from the functional integral is (5). We obtain

$$\left\langle \mathrm{e}^{\mathrm{i}\sum p_i\varphi(x_i)} \right\rangle = \exp\left( -\frac{1}{2} \sum_{i,j}^{N} p_ip_j \langle \varphi(x_i)\varphi(x_j) \rangle \right) = \prod_{i,j}^{N} \left( \frac{(z_i - z_j)(\bar{z}_i - \bar{z}_j)}{R^2} \right)^{p_ip_j}.$$

There are two types of terms in the exponent. For $i \neq j$ we may apply standard formula for pair correlation function, but if $i = j$ it formally gives infinity. Let us cut it at a small radius $r_0$:

$$\langle \varphi(x')\varphi(x) \rangle = \begin{cases} 2 \log \frac{R^2}{(z'-z)(\bar{z}'-\bar{z})}, & \text{if } |x^2| > r_0^2; \\ 2 \log \frac{R^2}{r_0^2}, & \text{if } |x^2| \leq r_0^2. \end{cases} \tag{9}$$

Then we have (if $|(x_i - x_j)^2| > r_0^2$)

$$\left\langle \mathrm{e}^{\mathrm{i}\sum p_i\varphi(x_i)} \right\rangle = \left( \frac{r_0^2}{R^2} \right)^{\sum p_i^2} \prod_{i>j}^{N} \left( \frac{(z_i - z_j)(\bar{z}_i - \bar{z}_j)}{R^2} \right)^{2p_ip_j}$$

$$= r_0^{2\sum p_i^2} R^{-2(\sum p_i)^2} \prod_{i>j}^{N} (z_i - z_j)^{2p_ip_j} (\bar{z}_i - \bar{z}_j)^{2p_ip_j}.$$

We see that the contribution of $r_0$ can be factorized between the exponents. Hence, the answer will coincide with (7), if we set

$$\mathrm{e}^{\mathrm{i}p\varphi(z)} = r_0^{2p^2} :\mathrm{e}^{\mathrm{i}p\varphi(z)}: . \tag{10}$$

It means that the normal ordered exponent $:\mathrm{e}^{\mathrm{i}p\varphi(z)}:$ is a proper renormalization of the true exponent $\mathrm{e}^{\mathrm{i}p\varphi(z)}$. This argument also explains why the normal ordered exponent possesses the scaling dimension $d = 2p^2$: any correlation function on an infinite plane is invariant under the substitution

$$V_p(x) \to \lambda^{2p^2} V_p(x). \tag{11}$$

In other words the scaling dimension $d$ is the eigenvalue of the dilation operator $D$ acting on the corresponding state $|p\rangle$.

Now we will be interested in the energy-momentum tensor. According to the usual formula in the Minkowski space

$$T^\mu_\nu = \partial_\nu \varphi \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} - \delta^\mu_\nu \mathcal{L}$$

we obtain

$$T_{zz} \equiv -2\pi T = \frac{1}{8\pi}(\partial\varphi)^2, \qquad T_{\bar{z}\bar{z}} \equiv -2\pi \bar{T} = \frac{1}{8\pi}(\bar{\partial}\varphi)^2, \qquad T_{z\bar{z}} \equiv 2\pi\Theta = 0. \tag{12}$$

The last equation reflects conformal invariance of the theory. The energy-momentum conservation law $\partial_\mu T^\mu_\nu = 0$ leads to

$$\bar{\partial} T = \partial \Theta, \qquad \partial \bar{T} = \bar{\partial} \Theta.$$

Since $\Theta = 0$ they reduce to

$$\bar\partial T = \partial \bar T = 0.$$

It means that the operator $T = T(z)$ is a function of the only variable $z$, while $\bar T = \bar T(\bar z)$ is a function of the only variable $\bar z$. On the Euclidean plane $T(z)$ is a complex analytic (holomorphic) function of $z$, while $\bar T(\bar z)$ is a holomorphic function of $\bar z$ (it is usually called 'antiholomorphic'). In quantum mechanics it is natural to define them as

$$T(z) = -\frac{1}{4} :(\partial\varphi)^2:, \qquad \bar T(\bar z) = -\frac{1}{4} :(\bar\partial\varphi)^2:. \tag{13}$$

On the cylinder it is natural to define the quantum energy-momentum tensor components according to

$$T_{\rm cyl}(\zeta) = -\left(\frac{2\pi}{L}\right)^2 \left(z^2 T(z) - \frac{1}{24}\right), \qquad \bar T_{\rm cyl}(\bar\zeta) = -\left(\frac{2\pi}{L}\right)^2 \left(\bar z^2 \bar T(z) - \frac{1}{24}\right). \tag{14}$$

The term $-1/24$ was added artificially to reproduce the term $-\pi/6L$ in the Hamiltonian $H$ being written in terms of the energy-momentum tensor:

$$H = \int_0^L d\xi^1\, T^{00}(\xi) = -\int_0^L \frac{d\xi^1}{2\pi} (T_{\rm cyl}(\xi^1) + \bar T_{\rm cyl}(\xi^1)). \tag{15}$$

In one of the next lectures we will understand the true origin of this term.

It is convenient to introduce the Laurent components of the energy-momentum tensor:

$$\begin{aligned} L_k &= \oint \frac{dz}{2\pi {\rm i}} z^{k+1} T(z) = p^2 \delta_{k0} + \frac{1}{4} \sum_{l\in\mathbb{Z}\setminus\{0,k\}} :\alpha_l \alpha_{k-l}:, \\ \bar L_k &= \oint \frac{d\bar z}{2\pi {\rm i}} \bar z^{k+1} T(\bar z) = p^2 \delta_{k0} + \frac{1}{4} \sum_{l\in\mathbb{Z}\setminus\{0,k\}} :\bar\alpha_l \bar\alpha_{k-l}:. \end{aligned} \tag{16}$$

The integration contours are defined in the Euclidean plane. They encircle coordinate origin $z = 0$ ($\bar z = 0$) in the counter-clockwise direction of $z$ ($\bar z$) and all points, where operators, which stand to the right of $T(z)$ ($\bar T(\bar z)$), are placed. For example, in the product $T(u) L_k T(w)$ it will enclose $w$, but not $u$.

In these components (15) can be rewritten as

$$H = \frac{2\pi}{L}\left(L_0 + \bar L_0 - \frac{1}{12}\right). \tag{17}$$

Similarly, the momentum operator on the cylinder is given by

$$P = \frac{2\pi}{L}(L_0 - \bar L_0). \tag{18}$$

On the plane the operator $D = L_0 + \bar L_0$ is the dilation operator, while $S = L_0 - \bar L_0$ is the angular momentum operator. Translations are generated by the operators $L_{-1}, \bar L_{-1}$. We may set

$$H_{\rm plane} = -{\rm i}(L_{-1} - \bar L_{-1}), \qquad P_{\rm plane} = -{\rm i}(L_{-1} + \bar L_{-1}). \tag{19}$$

Consider the operator product

$$\begin{aligned} T(z')T(z) &= \frac{1}{16} :(\partial\varphi(x'))^2: \,:(\partial\varphi(x))^2: \\ &= \frac{1}{16} :(\partial\varphi(x'))^2 (\partial\varphi(x))^2: + \frac{1}{4} \langle \partial\varphi(x')\, \partial\varphi(x)\rangle :\partial\varphi(x')\, \partial\varphi(x): + \frac{1}{8} \langle \partial\varphi(x')\, \partial\varphi(x)\rangle^2. \end{aligned}$$

We want to count singularities of the expression. The normal products are regular as $z' \to z$. Hence, the only singularity can stem from the correlation function

$$\langle \partial\varphi(x')\, \partial\varphi(x)\rangle = -\frac{\partial^2}{\partial z'^2} \langle \varphi(x')\varphi(x)\rangle = \frac{2}{(z'-z)^2}.$$

By expanding everything in powers of $z'-z$ we obtain the following operator product expansion (OPE):

$$T(z')T(z)=\frac{1/2}{(z'-z)^4}+\frac{T(z)}{(z'-z)^2}+\frac{\partial T(z)}{z'-z}+O(1). \tag{20}$$

Consider now the commutator

$$[L_k,L_l]=L_kL_l-L_lL_k=\oint_{C_{\text{out}}}\frac{dz'}{2\pi\mathrm{i}}\oint_C\frac{dz}{2\pi\mathrm{i}}z'^{k-1}z^{l-1}T(z')T(z)-\oint_C\frac{dz}{2\pi\mathrm{i}}\oint_{C_{\text{in}}}\frac{dz'}{2\pi\mathrm{i}}z'^{k-1}z^{l-1}T(z')T(z).$$

The two terms differ in the order of contours. The contour $C_{\text{out}}$ encloses the contour $C$, while the contour $C_{\text{in}}$ is enclosed by $C$. Hence,

$$[L_k,L_l]=\oint_C\frac{dz}{2\pi\mathrm{i}}\oint_{C_z}\frac{dz'}{2\pi\mathrm{i}}z'^{k-1}z^{l-1}T(z')T(z).$$

Here $C_z$ is a small contour that encloses $z$ in a counter-clockwise direction. After substituting the OPE (20) and evaluating the residue we obtain

$$[L_k,L_l]=(k-l)L_{k+l}+\frac{1}{12}k(k^2-1)\delta_{k+l,0}. \tag{21}$$

The algebra (21) is called the *Virasoro algebra*. The same is valid for the left (antichiral) component $\bar T(\bar z)$, which produces another copy of the Virasoro algebra.

Similarly one can prove the OPE

$$T(z')V_p(x)=\frac{p^2V_p(x)}{(z'-z)^2}+\frac{\partial V_p(x)}{z'-z}+O(1). \tag{22}$$

As a commutator one can rewrite it as

$$[L_k,V_p(x)]=p^2(k+1)z^kV_p(x)+z^{k+1}\partial V_p(x). \tag{23}$$

The sense of the above equations will be clarified later, when we will study conformal field theory.

### Problems

**1.** Accurately prove the formulas (3), (4).

**2.** Let us modify the energy-momentum tensor as follows:[1]

$$T(z)=-\frac14(\partial\varphi)^2+\frac{\mathrm{i}\mathcal{Q}}{2}\partial^2\varphi,\qquad \bar T(\bar z)=-\frac14(\bar\partial\varphi)^2+\frac{\mathrm{i}\mathcal{Q}}{2}\bar\partial^2\varphi. \tag{24}$$

Prove that

$$T(z')T(z)=\frac{c/2}{(z'-z)^4}+\frac{T(z)}{(z'-z)^2}+\frac{\partial T(z)}{z'-z}+O(1),\qquad c=1-6\mathcal{Q}^2,$$

and

$$[L_k,L_l]=(k-l)L_{k+l}+\frac{c}{12}k(k^2-1)\delta_{k+l,0}.$$

**3.** Prove that for $T(z)$ from (24) the Virasoro algebra generators read

$$L_k=p^2\delta_{k0}+\frac14\sum_{l\in\mathbb{Z}\setminus\{0,k\}}{:}\alpha_l\alpha_{k-l}{:}+\frac{k+1}{2}\mathcal{Q}\alpha_k.$$

**4.** For $T(z)$ from (24) prove the OPE

$$T(z')V_p(x)=\frac{\Delta_pV_p(x)}{(z'-z)^2}+\frac{\partial V_p(x)}{z'-z}+O(1),\qquad \Delta_p=p(p-\mathcal{Q}).$$

[1] We will see below that this form of the energy-momentum tensor is consistent with the Liouville theory, if $\mathrm{i}\mathcal{Q}=b+b^{-1}$.

**5.** Define the so called correlation functions with a charge at infinity:

$$\langle X\rangle_{\mathcal{Q}} = \lim_{x_0\to\infty}(z_0\bar{z}_0)^{2\mathcal{Q}^2}\langle XV_{-\mathcal{Q}}(x_0)\rangle.$$

Prove that for exponential operators they read

$$\left\langle \prod_{i=1}^{\widehat{N}} V_{p_i}(x_i)\right\rangle_{\mathcal{Q}} = \prod_{i>j}^{N}(z_i-z_j)^{2p_ip_j}(\bar{z}_i-\bar{z}_j)^{2p_ip_j}\times\begin{cases}1, & \sum_{i=1}^N p_i=\mathcal{Q};\\ 0, & \sum_{i=1}^N\neq 0.\end{cases} \tag{25}$$

Show the functions (25) to be invariant under the transformation

$$V_p(z,\bar{z})\to(f'(z))^{\Delta_p}(\bar{f}'(\bar{z}))^{\Delta_p}V_p(f(z),\bar{f}(\bar{z})),$$

if $f(z)$ is a Möbius transformation

$$f(z)=\frac{az+b}{cz+d}.$$