Lecture 8
Heisenberg spin chain and its scaling limit

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$$H_{XYZ} = -\frac{1}{2} \sum_{n=1}^{N} \left(J_x \sigma_n^x \sigma_{n+1}^x + J_y \sigma_n^y \sigma_{n+1}^y + J_z \sigma_n^z \sigma_{n+1}^z \right) \tag{2}$$

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- XYZ chain: generic J_i;
- XXZ chain: $|J_x| = |J_y|$;
- XXX chain: $|J_x| = |J_y| = |J_z|$;
- XY chain: $J_z = 0$. Today we will mostly consider this case.

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where

$$\sigma^+ = \frac{\sigma^x + i\sigma^y}{2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \sigma^- = \frac{\sigma^x - i\sigma^y}{2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{4}$$

Jordan–Wigner transformation

The matrices $\sigma^{\pm}$ behave like fermions

$$\sigma^{+}\sigma^{-} + \sigma^{-}\sigma^{+} = 1, \qquad (\sigma^{+})^2 = (\sigma^{-})^2 = 0.$$

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It is invertible

$$\sigma^z_n = 2a^+_n a_n - 1, \qquad \sigma^+_n = a^+_n \exp\left(i\pi \sum_{j=1}^{n-1} a^+_j a_j\right), \qquad \sigma^-_n = a_n \exp\left(-i\pi \sum_{j=1}^{n-1} a^+_j a_j\right). \tag{6}$$

Jordan–Wigner transformation 2

The price payed for the transformation is a small change of the boundary condition:

$$a_{N+1} = a_1(-1)^M, \qquad M = \sum_{k=1}^{N} a_k^+ a_k = S^z + \frac{N}{2}, \tag{7}$$

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When is this Hamiltonian solvable? Evidently, for $\Delta = 0$.

XY model. Bogoliubov transform: XX case

Set $\Delta = 0$. Let us pass to the momentum space

$$a_n = \frac{1}{N^{1/2}} \sum_{-\pi < k \leq \pi} a_k e^{ikn}, \quad \frac{kN}{2\pi} \in \begin{cases} \mathbb{Z}, & M \text{ even;} \\ \mathbb{Z} + \frac{1}{2}, & M \text{ odd.} \end{cases} \tag{9}$$

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$$H_{XY} = - \sum_{-\pi < k \leq \pi} \left((1+\Gamma) \cos k \cdot a_k^+ a_k + i \frac{1-\Gamma}{2} \sin k \cdot (a_k^+ a_{-k}^+ + a_k a_{-k}) \right). \tag{10}$$

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- for $|k| > k_F = \frac{\pi}{2}$ we have $\epsilon_k^{(0)} > 0$;
- for $|k| < k_F = \frac{\pi}{2}$ we have $\epsilon_k^{(0)} < 0$.

The last region must be filled up in the ground state ('Dirac sea').

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Since the space of states is finite-dimensional, we may redefine the oscillators:

$$b_k = a_{k-\pi}, \quad b_k^+ = a_{k-\pi}^+, \quad b'_k = i a_k^+, \quad b'^+_k = -i a_k, \quad -\frac{\pi}{2} < k \leq \frac{\pi}{2}, \tag{11}$$

This is the simplest Bogoliubov transform.

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$$H_{XX} = \sum_{-\pi/2<k\le\pi/2} \epsilon_k (b_k^+ b_k + b_k'^+ b'_k), \quad \epsilon_k = \cos k \ge 0.$$

But there are particles b_k and antiparticles b'_k.

XY model. Bogoliubov transform: general case

Now return to generic Γ. Consider the Bogoliubov transform

$$\begin{aligned} a_{k-\pi} &= \alpha_k b_k + \beta_k b^+_{-k}, \quad a^+_{k-\pi} = \beta'^*_k b_{-k} + \alpha'^*_k b^+_k, \\ a_k &= \alpha'_k b'_k + \beta'_k b'^+_{-k}, \quad a^+_k = \beta'^*_k b'_{-k} + \alpha'^*_k b'^+_k, \end{aligned} \tag{12}$$

where

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Now we should adjust the coefficients $\alpha_k, \ldots$ to eliminate terms of the form bb, b^+b^+, $b'b'$, $b'^+b'^+$ in the Hamiltonian. We obtain

$$\begin{aligned} \alpha_k &= \cos\frac{\kappa}{2}, \qquad \beta_k = i\sin\frac{\kappa}{2}, \\ \alpha'_k &= -\sin\frac{\kappa}{2}, \quad \beta'_k = i\cos\frac{\kappa}{2}, \end{aligned} \qquad \operatorname{tg}\kappa = \frac{1-\Gamma}{1+\Gamma}\operatorname{tg}k. \tag{13}$$

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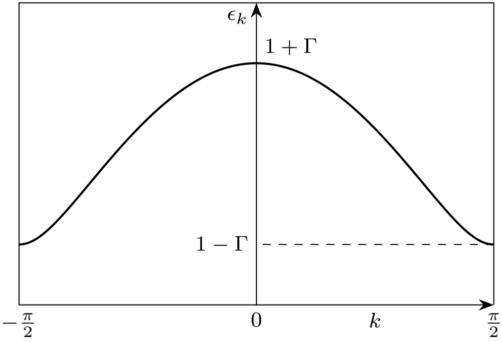
The Hamiltonian takes the form

$$H_{XY} = \sum_{-\pi/2<k\le\pi/2} \epsilon_k (b^+_k b_k + b'^+_k b'_k), \quad \epsilon_k = \sqrt{(1+\Gamma)^2\cos^2 k + (1-\Gamma)^2 \sin^2 k}. \tag{14}$$

XY model. Spectrum

$$\epsilon_k = \sqrt{(1+\Gamma)^2 \cos^2 k + (1-\Gamma)^2 \sin^2 k}$$



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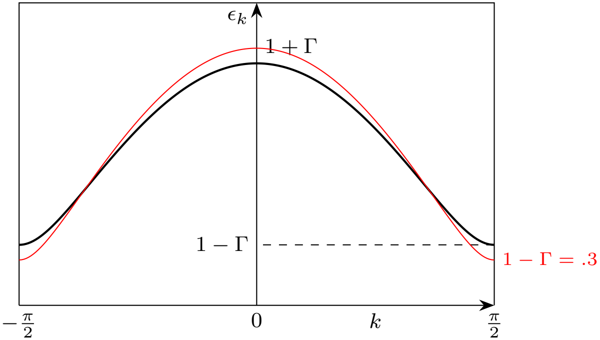

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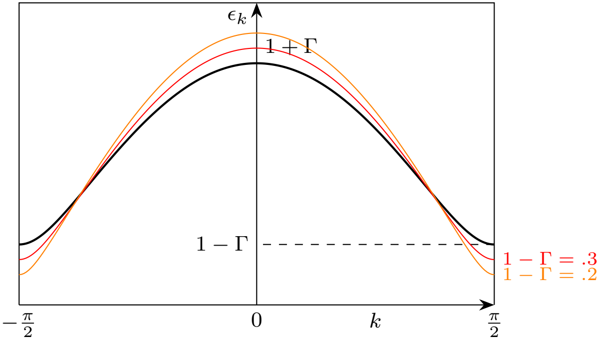

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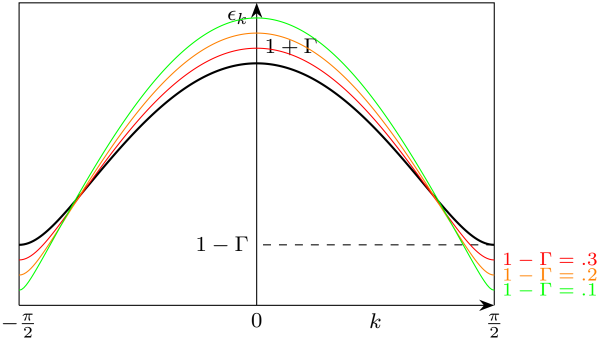

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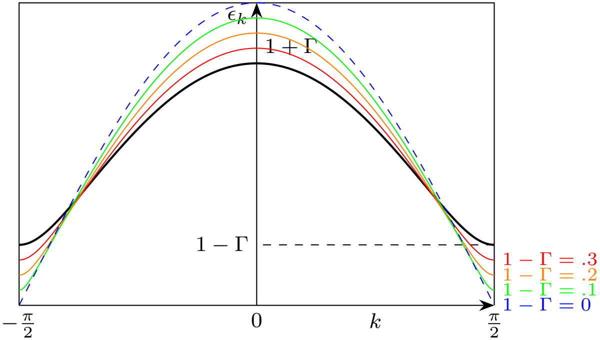

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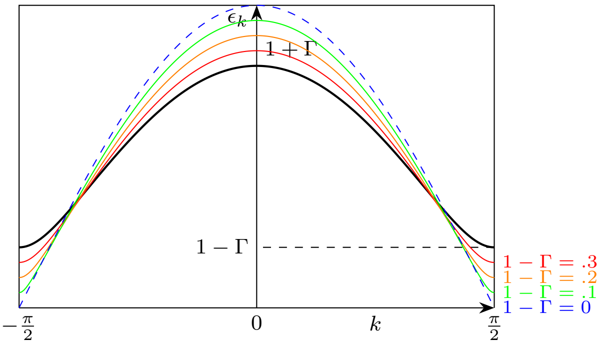

The minimum of ϵ_k is equal to

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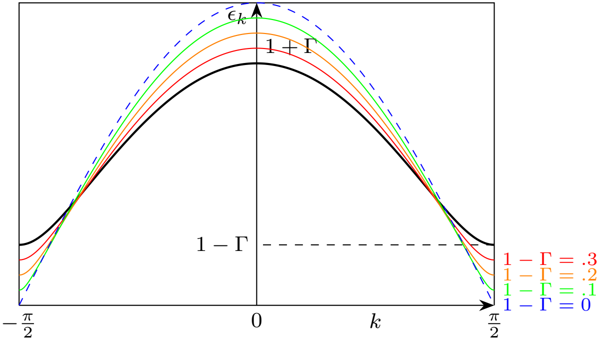

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- $1-\Gamma$ and achieved at $k = \frac{\pi}{2}$ for $\Gamma > 0$;
- $1+\Gamma$ and achieved at $k = 0$ for $\Gamma < 0$.

Hence, the spectrum has a mass gap of $2\min(1+\Gamma, 1-\Gamma)$. In the limits $\Gamma \to \pm 1$ the gap disappears and the system admits scaling limit. Without loss of generality we will consider the limit $\Gamma \to 1$.

XY model. Scaling limit

Let $1 - \Gamma \ll 1$. Consider low-lying excitations. Let

$$pa = \begin{cases} \frac{\pi}{2} - k, & k > 0; \\ -\frac{\pi}{2} - k, & k < 0. \end{cases}$$

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Then

$$\epsilon(p) = 2a\sqrt{m^2 + p^2}, \qquad m = \frac{1-\Gamma}{2a}. \tag{15}$$

Up to a factor $2a$ this is the spectrum of a free relativistic particle. So define

$$H_{FF} = \frac{1}{2a} H_{XY}.$$

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The parameter of the Bogoliubov transform

$$\operatorname{ctg} \kappa = \frac{|p|}{m},$$

Consider the case $m = 0$ ($\Gamma = 1$ or XX model). The Bogoliubov transform becomes trivial

$$\begin{aligned} a_{\pi/2-pa} &= ib'^{+}_{-\pi/2+pa}, & a_{\pi/2+pa} &= b_{-\pi/2+pa}, \\ a_{-\pi/2+pa} &= ib'^{+}_{\pi/2-pa}, & a_{-\pi/2-pa} &= b_{\pi/2-pa}. \end{aligned} \tag{16}$$

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Introduce the operators

$$\psi_{\pm}(x) = \int_{-\infty}^{\infty} \frac{dp}{2\pi}\, a_{\pm}(p) e^{ipx} \qquad a_{\pm}(p) = (Na)^{1/2} a_{\pm\pi/2+pa}, \tag{17}$$

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The initial fermions a_n are expressed in terms of them as

$$a_n = a^{1/2}(i^n \psi_{+}(an) + i^{-n}\psi_{-}(an)).$$

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Define

$$H_\Delta \equiv \frac{1}{2a}(H_{XXZ} + \Delta N/2 - 2\Delta M) = H_{FF} - \frac{\Delta}{Na}\sum_q \rho_q \rho_{-q} \cos q, \qquad (18)$$

where

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Scaling XXZ model

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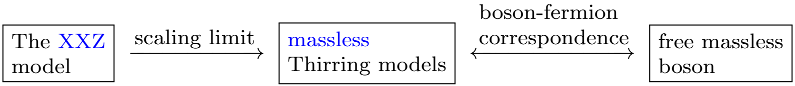
The exact relation for finite Δ is known from an exact solution of the XYZ model:

$$\beta^2 = \frac{2\mu}{\pi}, \quad \frac{g}{\pi} = \frac{\pi/2 - \mu}{\mu}, \quad \Delta = -\cos\mu. \qquad (22)$$

XYZ model and sine-Gordon model



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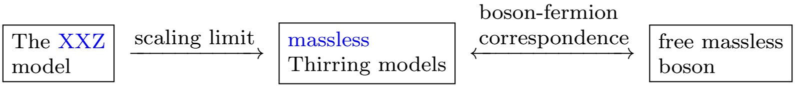
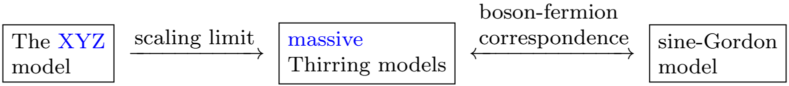


Conjecture:



XYZ model and sine-Gordon model: symmetries

Symmetry of the free massless boson:

$$\varphi \to \varphi + \alpha, \quad \alpha \in \mathbb{R}.$$

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Identification:

$$\alpha = \frac{2\lambda}{\beta} \tag{25}$$

Therefore

$$\sigma_n^{\pm}\sigma_{n+1}^{\pm} \sim a^{\beta^2} e^{\pm i\beta\phi}.$$

XYZ model and sine-Gordon model: identification of operators

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It can be conjectured that

$$\sigma_n^\pm \sim a^{\beta^2/4} e^{\pm i\frac{\beta}{2}\phi}.$$

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What is σ^z? A more accurate study gives

$$\sigma_n^z = c_1 a\partial_t\phi + c_2(-1)^n a^{1/\beta^2}\sin\frac{i\tilde{\phi}}{\beta}.$$