Lecture 1.
$O(2)$-model and Berezinskii–Kosterlitz–Thouless transition

Michael Lashkevich

$O(2)$-model

We will often consider the models in two-dimensional space-time with the action

$$S[\boldsymbol{n}] = \frac{1}{2g}\int d^2x\,(\partial_\mu \boldsymbol{n})^2, \qquad \boldsymbol{n}^2 \equiv \sum_{i=1}^{N} n_i^2 = 1, \tag{1}$$

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What can break this behavior?

Vortices in the Euclidean plane

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Then for any smooth, bounded and decreasing fast enough function $\varphi(x)$ we have

$$\int d^2x\,\varphi(x)\partial_\mu\partial^\mu\frac{1}{2i}\log\frac{z}{\bar z} = \int d^2x\,(\epsilon^{\mu\nu}\partial_\mu\partial_\nu\varphi(x))\log\frac{1}{r} = 0,$$

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$$\int d^2x\, \partial^\mu\varphi\, \partial_\mu\varphi_{\vec q\vec x} = 0. \tag{7}$$

Classical action of vertices

Let us calculate the classical action on the vertex solution:

$$S[\varphi_{\vec{q}\vec{x}}] = \frac{2}{g} \int d^2x \, \partial\varphi_{\vec{q}\vec{x}} \, \bar{\partial}\varphi_{\vec{q}\vec{x}}$$

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The first integral is

$$\int \frac{d^2x}{|z - z_a|^2} \simeq 2\pi \int_{r_0}^{R} \frac{dr}{r} = 2\pi \log \frac{R}{r_0},$$

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Hence

$$S[\varphi_{\vec q\vec x}] = \frac{1}{2g}\left(\pi\sum_a q_a^2\log\frac{R^2}{r_0^2} + 2\pi\sum_{a<b} q_a q_b \log\frac{R^2}{|z_a-z_b|^2}\right) \tag{8}$$

$$= \frac{\pi}{2g}\left(\sum_a q_a\right)^2 \log R^2 - \frac{\pi}{2g}\sum_a q_a^2\log r_0^2 + \frac{1}{2g}\sum_{a<b} q_a q_b\, 2\pi\log\frac{1}{|z_a-z_b|^2}. \tag{9}$$

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We will see that the behavior of the gas of vortices depends on g rather than on r_0.

Functional integral

Now we want to calculate the functional integral over φ. We split it into a sum over vortex configurations:

$(f,g)=\int d^2x\, f(x)g(x)$

$$Z[J]=\sum_{n=0}^{\infty}\frac{r_0^{-2n}}{n!}\sum_{\substack{q_1,\ldots,q_n\neq 0\\ q_1+\cdots+q_n=0}}\int d^2x_1\cdots d^2x_n\int D\chi\, e^{-S[\chi+\varphi_{\vec q\vec x}]-(J,\chi+\varphi_{\vec q\vec x})}. \quad (11)$$

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Here n is the number over vortices, q_i are vorticities, x_i are position of vortices, and the field χ runs all configurations without the identification ~~$\chi \sim \chi + 2\pi$~~. The factor $1/n!$ is caused by the fact that configurations of vortices are permutation invariant. The factor r_0^{-2n} is necessary to nondimensionolize the integrals. We may think that every vortex lives in a cell of size r_0.

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We have seen that the last term vanishes. Hence the generating function factorizes:

$$Z[J]=Z_0[J]\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{\substack{q_1,\dots,q_n\\ q_1+\dots+q_n=0}} r_0^{\frac{\pi}{g}\sum q_a^2-2n}\times$$

$$\times\int d^2x_1\cdots d^2x_n\left(\prod_{a<b}|z_a-z_b|^{2\frac{\pi}{g}q_aq_b}\right)e^{-(J,\varphi_{\vec q\vec x})}, \quad (12)$$

$\left(\prod_{a<b}\dots\right) = e^{-S[\varphi_{\vec q\vec x}]}\times\text{const}$

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Berezinskii–Kosterlitz–Thouless (BKT) transition

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This looks as a partition function of a two-dimensional plasma. At high ‘temperature’ g the plasma is ‘ionized’, vortices are separated and correlation functions decrease exponentially due to the Debye-type screening. Al low ‘temperature’ the vortices of opposite vorticities attract and neutralize each other. In contrast to the usual plasma here these regimes are switched at a definite value of g. It is called the Berezinskii–Kosterlitz–Thouless (BKT) transition.

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Now let us find the critical value g_{BKT} of the 'temperature' g corresponding to the BKT transition.

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For large n it gives the exact bound: if this condition is satisfied, all integrals are convergent, while if it is not satisfied, there exists a configuration $\{q_a\}$ for any given n, for which the integral diverges. By taking $n \to \infty$ we obtain the critical value

$$g_{\text{BKT}} = \frac{\pi}{2}. \tag{16}$$

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- For $g < g_{\rm BKT}$ the theory is massless and for $r \gg r_0$ coincides with a reduction of the free massless boson theory compatible with the identification $\varphi \sim \varphi + 2\pi$.

Free massless field

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We rewrite it as follows

$$\begin{aligned} \phi(x) &= \phi_R(z) + \phi_L(\bar{z}), \\ \tilde{\phi}(x) &= \phi_R(z) - \phi_L(\bar{z}). \end{aligned} \tag{20}$$

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This decomposition (up to some subtleties) is valid in the quantum case. The correlation functions

$$\langle \phi_R(z) \phi_R(z') \rangle_0 = \log \frac{R}{z - z'}, \quad \langle \phi_L(\bar{z}) \phi_L(\bar{z}') \rangle_0 = \log \frac{R}{\bar{z} - \bar{z}'}, \quad \langle \phi_R(z) \phi_L(\bar{z}') \rangle_0 = 0 \tag{21}$$

are consistent with the theory.

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$$=\exp\left(-\frac12\sum_{a=1}^n\alpha_a^2\overbrace{\langle\phi_R^2\rangle_0}^{\log\frac{R}{r_0}}-\sum_{a<b}\alpha_a\alpha_b\overbrace{\langle\phi_R(z_a)\phi(z_b)\rangle_0}^{\log\frac{R}{z_a-z_b}}\right)=\left(\frac{r_0}{R}\right)^{\frac12\sum_a\alpha_a^2}\prod_{a<b}^n\left(\frac{z_a-z_b}{R}\right)^{\alpha_a\alpha_b}$$

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$$e^{i\alpha\phi_{R,L}} = r_0^{\alpha^2/2} :e^{i\alpha\phi_{R,L}}:, \qquad e^{i\alpha\phi} = r_0^{\alpha^2} :e^{i\alpha\phi}:, \qquad e^{i\alpha\tilde\phi} = r_0^{i\alpha^2} :e^{i\alpha\tilde\phi}:. \qquad (22)$$

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Indeed, in the limit $R \to \infty$ we have

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These correlation functions are invariant under the scaling transformation. Then we have

$$\left\langle \prod_{j=1}^{k} e^{i\beta_j\tilde\phi(y_j)} \prod_{a=1}^{n} e^{i\alpha_a\phi(x_a)} \right\rangle_0 = r_0^{\sum_a \alpha_a^2 + \sum_j \beta_j^2} \prod_{a<b} |z_a - z_b|^{2\alpha_a\alpha_b} \times$$
$$\times \prod_{j<j'} |w_j - w_{j'}|^{2\beta_a\beta_b} \prod_{a,j} \left(\frac{w_j - z_a}{\bar w_j - \bar z_a}\right)^{\alpha_a\beta_j} \times \begin{cases} 1, & \sum \alpha_a = \sum \beta_j = 0; \\ 0 & \text{otherwise}. \end{cases} \tag{25}$$

This coincides with the integrand of $Z[J]$ if

$$\alpha_a = \sqrt{\frac{\pi}{g}}\, q_a, \qquad \beta_j = \sqrt{\frac{g}{4\pi}}\, J_j. \tag{26}$$

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$$Z[J_{\vec{J}\vec{y}}] = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\substack{q_1,\dots,q_n \\ q_1+\dots+q_n=0}} r_0^{-2n} \int d^2x_1 \cdots d^2x_n \times \left\langle \prod_{j=1}^{k} e^{i\sqrt{\frac{g}{4\pi}}\, J_j \tilde{\phi}(y_j)} \prod_{a=1}^{n} e^{i\sqrt{\frac{\pi}{g}}\, q_a \phi(x_a)} \right\rangle_0 .$$

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The integrand is remarkably symmetric with respect to the replacements

$$g \leftrightarrow (2\pi)^2 g^{-1}, \qquad k \leftrightarrow n, \qquad q_a \leftrightarrow J_j, \qquad \phi(x) \leftrightarrow \tilde{\phi}(x).$$

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Moreover, the Lagrangian of the free field is written identically in terms of both the fields ϕ and $\tilde{\phi}$. Thus we can identify

$$\varphi(x) = \sqrt{\frac{g}{4\pi}}\tilde{\phi}(x). \tag{27}$$

Since $r_0^{\frac{\pi}{g}q^2} \ll r_0^{q\frac{\pi}{g}}$, we may neglect the contribution of a q-vortex compared to the contribution of q instances of a 1-vortex.

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&= \int D\phi \, e^{-S_{\mathrm{SG}}[\phi]} \prod_{j=1}^{k} e^{i\sqrt{\frac{g}{4\pi}} J_j \tilde{\phi}(y_j)},
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where

$$S_{\mathrm{SG}}[\phi] = \int d^2x \left(\frac{(\partial_\mu\phi)^2}{8\pi} - \mu :\cos\beta\phi: \right) \qquad (29)$$

is the action of the sine-Gordon model with the parameters

$$\beta = \sqrt{\frac{\pi}{g}}, \qquad \mu = 2r_0^{\frac{\pi}{g}-2}. \qquad (30)$$

Scaling dimension of the perturbation term

The sine-Gordon model is a perturbation of the free massless fermion model with the perturbation term $\sim\ :\cos\beta\phi$ in the Lagrangian with the scaling dimension

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3. $\Delta_{\mathrm{p}} = 2$ ($g = g_{\mathrm{BKT}}$). The perturbation is marginal. In the case of the sine-Gordon theory it is also renormalizable. Nevertheless it changes both infrared and ultraviolet behavior.

1. Define

$$\varphi(z) = Q - iP\log z + \sum_{k\neq 0} \frac{a_k}{ik} z^{-k},$$

$$[P,Q] = -i, \qquad [a_k, a_l] = k\delta_{k+l,0},$$

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Seminar: free massless boson

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$$e^{i\alpha\varphi(r_0,z)} = \exp\left(i\alpha Q + \alpha P\log z + \alpha\sum_{k>0}\left(\frac{a_k}{k}z^{-k} - \frac{a_{-k}}{k}(z-r_0)^k\right)\right),$$

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$$\langle\varphi(z')\varphi(z)\rangle = \langle Q^2\rangle + \log\frac{1}{z'-z}.$$

2. Define

$$e^{i\alpha\varphi(r_0,z)} = \exp\left(i\alpha Q + \alpha P\log z + \alpha\sum_{k>0}\left(\frac{a_k}{k}z^{-k} - \frac{a_{-k}}{k}(z-r_0)^k\right)\right),$$

$$:e^{i\alpha\varphi(r_0,z)}: = e^{i\alpha Q}z^{\alpha P}\exp\left(-\alpha\sum_{k>0}\frac{a_{-k}}{k}(z-r_0)^k\right)\exp\left(\alpha\sum_{k>0}\frac{a_k}{k}z^{-k}\right).$$

Calculate the coefficient:

$$e^{i\alpha\varphi(r_0,z)} = {\color{red}??} \qquad :e^{i\alpha\varphi(r_0,z)}:.$$

1. Define

$$\varphi(z) = Q - iP\log z + \sum_{k\neq 0}\frac{a_k}{ik}z^{-k},$$

$$[P,Q] = -i, \qquad [a_k,a_l] = k\delta_{k+l,0},$$

$$P|0\rangle = a_k|0\rangle = 0, \quad \langle 0|a_{-k} = 0 \quad (k>0).$$

Calculate

$$\langle\varphi(z')\varphi(z)\rangle = \langle Q^2\rangle + \log\frac{1}{z'-z}.$$

2. Define

$$e^{i\alpha\varphi(r_0,z)} = \exp\left(i\alpha Q + \alpha P\log z + \alpha\sum_{k>0}\left(\frac{a_k}{k}z^{-k} - \frac{a_{-k}}{k}(z-r_0)^k\right)\right),$$

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Calculate the coefficient:

$$e^{i\alpha\varphi(r_0,z)} = r_0^{\alpha^2/2}:e^{i\alpha\varphi(r_0,z)}:.$$

3. Define

$$:e^{i\alpha\varphi(z)}: = :e^{i\alpha\varphi(0,z)}: = e^{i\alpha Q} z^{\alpha P} \exp\left(-\alpha \sum_{k>0} \frac{a_{-k}}{k} z^k\right) \exp\left(\alpha \sum_{k>0} \frac{a_k}{k} z^{-k}\right).$$

3. Define

$$:e^{i\alpha\varphi(z)}: = :e^{i\alpha\varphi(0,z)}: = e^{i\alpha Q} z^{\alpha P} \exp\left(-\alpha \sum_{k>0} \frac{a_{-k}}{k} z^k\right) \exp\left(\alpha \sum_{k>0} \frac{a_k}{k} z^{-k}\right).$$

Calculate the coefficient

$$:e^{i\alpha_1\varphi(z')}: :e^{i\alpha_2\varphi(z)}: = ?? \qquad :e^{i\alpha_2\varphi(z')+i\alpha_1\varphi(z)}:.$$

3. Define

$$:e^{i\alpha\varphi(z)}: = :e^{i\alpha\varphi(0,z)}: = e^{i\alpha Q} z^{\alpha P} \exp\left(-\alpha \sum_{k>0} \frac{a_{-k}}{k} z^k\right) \exp\left(\alpha \sum_{k>0} \frac{a_k}{k} z^{-k}\right).$$

Calculate the coefficient

$$:e^{i\alpha_1\varphi(z')}: :e^{i\alpha_2\varphi(z)}: = (z'-z)^{\alpha_1\alpha_2} :e^{i\alpha_2\varphi(z')+i\alpha_1\varphi(z)}:.$$

3. Define

$$:e^{i\alpha\varphi(z)}: = :e^{i\alpha\varphi(0,z)}: = e^{i\alpha Q} z^{\alpha P} \exp\left(-\alpha \sum_{k>0} \frac{a_{-k}}{k} z^k\right) \exp\left(\alpha \sum_{k>0} \frac{a_k}{k} z^{-k}\right).$$

Calculate the coefficient

$$:e^{i\alpha_1\varphi(z')}: :e^{i\alpha_2\varphi(z)}: = (z'-z)^{\alpha_1\alpha_2} :e^{i\alpha_2\varphi(z')+i\alpha_1\varphi(z)}:.$$

4. Calculate

$$\langle :e^{i\sum_{j=1}^N \alpha_i\varphi(z_i)}: \rangle = ??$$

3. Define

$$:e^{i\alpha\varphi(z)}: = :e^{i\alpha\varphi(0,z)}: = e^{i\alpha Q} z^{\alpha P} \exp\left(-\alpha \sum_{k>0} \frac{a_{-k}}{k} z^k\right) \exp\left(\alpha \sum_{k>0} \frac{a_k}{k} z^{-k}\right).$$

Calculate the coefficient

$$:e^{i\alpha_1\varphi(z')}: :e^{i\alpha_2\varphi(z)}: = (z'-z)^{\alpha_1\alpha_2} :e^{i\alpha_2\varphi(z')+i\alpha_1\varphi(z)}:.$$

4. Calculate

$$\langle :e^{i\sum_{j=1}^N \alpha_i\varphi(z_i)}: \rangle = \begin{cases} 1, & \sum_j \alpha_j = 0; \\ 0 & \text{otherwise.} \end{cases}$$

3. Define

$$:e^{i\alpha\varphi(z)}: = :e^{i\alpha\varphi(0,z)}: = e^{i\alpha Q} z^{\alpha P} \exp\left(-\alpha \sum_{k>0} \frac{a_{-k}}{k} z^k\right) \exp\left(\alpha \sum_{k>0} \frac{a_k}{k} z^{-k}\right).$$

Calculate the coefficient

$$:e^{i\alpha_1\varphi(z')}: :e^{i\alpha_2\varphi(z)}: = (z'-z)^{\alpha_1\alpha_2} :e^{i\alpha_2\varphi(z')+i\alpha_1\varphi(z)}:.$$

4. Calculate

$$\langle :e^{i\sum_{j=1}^N \alpha_i\varphi(z_i)}: \rangle = \begin{cases} 1, & \sum_j \alpha_j = 0; \\ 0 & \text{otherwise.} \end{cases}$$

5. Calculate

$$\langle :e^{i\alpha_1\varphi(z_1)}: \cdots :e^{i\alpha_N\varphi(z_N)}: \rangle = ??$$

3. Define

$$:e^{i\alpha\varphi(z)}: \,=\, :e^{i\alpha\varphi(0,z)}: \,=\, e^{i\alpha Q} z^{\alpha P} \exp\left(-\alpha \sum_{k>0} \frac{a_{-k}}{k} z^k\right) \exp\left(\alpha \sum_{k>0} \frac{a_k}{k} z^{-k}\right).$$

Calculate the coefficient

$$:e^{i\alpha_1\varphi(z')}: \, :e^{i\alpha_2\varphi(z)}: \,=\, (z'-z)^{\alpha_1\alpha_2} \, :e^{i\alpha_2\varphi(z')+i\alpha_1\varphi(z)}:.$$

4. Calculate

$$\langle :e^{i\sum_{j=1}^N \alpha_i\varphi(z_i)}: \rangle = \begin{cases} 1, & \sum_j \alpha_j = 0; \\ 0 & \text{otherwise.} \end{cases}$$

5. Calculate

$$\langle :e^{i\alpha_1\varphi(z_1)}: \, \cdots \, :e^{i\alpha_N\varphi(z_N)}: \rangle = \prod_{i<j}^{N} (z_i - z_j)^{\alpha_i\alpha_j} \times \begin{cases} 1, & \sum_j \alpha_j = 0; \\ 0 & \text{otherwise.} \end{cases}$$

3. Define

$$:e^{i\alpha\varphi(z)}: = :e^{i\alpha\varphi(0,z)}: = e^{i\alpha Q} z^{\alpha P} \exp\left(-\alpha \sum_{k>0} \frac{a_{-k}}{k} z^k\right) \exp\left(\alpha \sum_{k>0} \frac{a_k}{k} z^{-k}\right).$$

Calculate the coefficient

$$:e^{i\alpha_1\varphi(z')}: :e^{i\alpha_2\varphi(z)}: = (z'-z)^{\alpha_1\alpha_2} :e^{i\alpha_2\varphi(z')+i\alpha_1\varphi(z)}:.$$

4. Calculate

$$\langle :e^{i\sum_{j=1}^N \alpha_i\varphi(z_i)}: \rangle = \begin{cases} 1, & \sum_j \alpha_j = 0; \\ 0 & \text{otherwise.} \end{cases}$$

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$$\langle :e^{i\alpha_1\varphi(z_1)}: \cdots :e^{i\alpha_N\varphi(z_N)}: \rangle = \prod_{i<j}^{N} (z_i - z_j)^{\alpha_i\alpha_j} \times \begin{cases} 1, & \sum_j \alpha_j = 0; \\ 0 & \text{otherwise.} \end{cases}$$

6. Prove that this correlation function is invariant under the transformation

$$:e^{i\alpha_i\varphi(z_i)}: \to \lambda^{\alpha^2/2} :e^{i\alpha_i\varphi(\lambda z_i)}:.$$

3. Define

$$:e^{i\alpha\varphi(z)}: = :e^{i\alpha\varphi(0,z)}: = e^{i\alpha Q} z^{\alpha P} \exp\left(-\alpha \sum_{k>0} \frac{a_{-k}}{k} z^k\right) \exp\left(\alpha \sum_{k>0} \frac{a_k}{k} z^{-k}\right).$$

Calculate the coefficient

$$:e^{i\alpha_1\varphi(z')}: :e^{i\alpha_2\varphi(z)}: = (z'-z)^{\alpha_1\alpha_2} :e^{i\alpha_2\varphi(z')+i\alpha_1\varphi(z)}:.$$

4. Calculate

$$\langle :e^{i\sum_{j=1}^N \alpha_i\varphi(z_i)}: \rangle = \begin{cases} 1, & \sum_j \alpha_j = 0; \\ 0 & \text{otherwise.} \end{cases}$$

5. Calculate

$$\langle :e^{i\alpha_1\varphi(z_1)}: \cdots :e^{i\alpha_N\varphi(z_N)}: \rangle = \prod_{i<j}^{N} (z_i - z_j)^{\alpha_i\alpha_j} \times \begin{cases} 1, & \sum_j \alpha_j = 0; \\ 0 & \text{otherwise.} \end{cases}$$

6. Prove that this correlation function is invariant under the transformation

$$:e^{i\alpha_i\varphi(z_i)}: \to \lambda^{\alpha^2/2} :e^{i\alpha_i\varphi(\lambda z_i)}:.$$

7. Prove that this correlation function is invariant under the transformation

$$:e^{i\alpha_i\varphi(z_i)}: \to z_i^{-\alpha^2} :e^{i\alpha_i\varphi(-z_i^{-1})}:.$$