

Lecture 1. $O(2)$ -model and Berezinskii–Kosterlitz–Thouless transition

Michael Lashkevich

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$$S[\mathbf{n}] = \frac{1}{2g} \int d^2x (\partial_\mu \mathbf{n})^2, \quad \mathbf{n}^2 \equiv \sum_{i=1}^N n_i^2 = 1, \quad (1)$$

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What can break this behavior?

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$$\varphi_{q_1 x_1}(x) = q_1 \theta,$$

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Then for any smooth, bounded and decreasing fast enough function $\varphi(x)$ we have

$$\int d^2 x \varphi(x) \partial_\mu \partial^\mu \frac{1}{2i} \log \frac{z}{\bar{z}} = \int d^2 x (\epsilon^{\mu\nu} \partial_\mu \partial_\nu \varphi(x)) \log \frac{1}{r} = 0,$$

since the integral of $\log r$ converges at $x = 0$.

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since the integral of $\log r$ converges at $x = 0$. We immediately obtain

$$\int d^2 x \partial^\mu \varphi \partial_\mu \varphi_{\vec{q}\vec{x}} = 0. \quad (7)$$

Classical action of vertices

Let us calculate the classical action on the vertex solution:

$$S[\varphi_{\bar{q}\bar{x}}] = \frac{2}{g} \int d^2x \partial\varphi_{\bar{q}\bar{x}} \bar{\partial}\varphi_{\bar{q}\bar{x}}$$

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$$\int \frac{d^2x}{|z - z_a|^2} \simeq 2\pi \int_{r_0}^R \frac{dr}{r} = 2\pi \log \frac{R}{r_0},$$

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Hence

$$S[\varphi_{\bar{q}\bar{x}}] = \frac{1}{2g} \left(\pi \sum_a q_a^2 \log \frac{R^2}{r_0^2} + 2\pi \sum_{a < b} q_a q_b \log \frac{R^2}{|z_a - z_b|^2} \right) \quad (8)$$

$$= \frac{\pi}{2g} \left(\sum_a q_a \right)^2 \log R^2 - \frac{\pi}{2g} \sum_a q_a^2 \log r_0^2 + \frac{1}{2g} \sum_{a < b} q_a q_b 2\pi \log \frac{1}{|z_a - z_b|^2}. \quad (9)$$


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We will see that the behavior of the gas of vortices depends on g rather than on r_0 .

Functional integral

Now we want to calculate the functional integral over φ . We split it into a sum over vortex configurations:

$$Z[J] = \sum_{n=0}^{\infty} \frac{r_0^{-2n}}{n!} \sum_{\substack{q_1, \dots, q_n \neq 0 \\ q_1 + \dots + q_n = 0}} \int d^2x_1 \cdots d^2x_n \int D\chi e^{-S[\chi + \varphi \bar{q}\bar{x}] - (J, \chi + \varphi \bar{q}\bar{x})}. \quad (11)$$

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$$Z[J] = Z_0[J] \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\substack{q_1, \dots, q_n \\ q_1 + \dots + q_n = 0}} r_0^{\frac{\pi}{g} \sum q_a^2 - 2n} \times e^{-S[\varphi_{\vec{q}\vec{x}}]} \times \text{const} \\ \times \int d^2x_1 \cdots d^2x_n \left(\prod_{a < b} |z_a - z_b|^{2\frac{\pi}{g} q_a q_b} \right) e^{-(J, \varphi_{\vec{q}\vec{x}})}, \quad (12)$$

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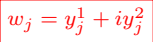
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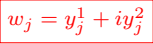
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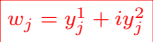
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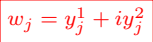
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$$g_{\text{BKT}} = \frac{\pi}{2}. \tag{16}$$

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- For $g < g_{\text{BKT}}$ the theory is massless and for $r \gg r_0$ coincides with a reduction of the free massless boson theory compatible with the identification $\varphi \sim \varphi + 2\pi$.

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This decomposition (up to some subtleties) is valid in the quantum case. The correlation functions

$$\langle \phi_R(z) \phi_R(z') \rangle_0 = \log \frac{R}{z - z'}, \quad \langle \phi_L(\bar{z}) \phi_L(\bar{z}') \rangle_0 = \log \frac{R}{\bar{z} - \bar{z}'}, \quad \langle \phi_R(z) \phi_L(\bar{z}') \rangle_0 = 0 \quad (21)$$

are consistent with the theory.

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Consider the exponents $e^{i\alpha\phi_{R,L}(x)}$ of the fields. Their correlation functions diverge. In the functional integral manner we can derive them as follows:

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$$e^{i\alpha\phi_{R,L}} = r_0^{\alpha^2/2} :e^{i\alpha\phi_{R,L}}:, \quad e^{i\alpha\phi} = r_0^{\alpha^2} :e^{i\alpha\phi}:, \quad e^{i\alpha\tilde{\phi}} = r_0^{i\alpha^2} :e^{i\alpha\tilde{\phi}}:. \quad (22)$$

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These correlation functions are invariant under the scaling transformation. Then we have

$$\begin{aligned} \left\langle \prod_{j=1}^k e^{i\beta_j\tilde{\phi}(y_j)} \prod_{a=1}^n e^{i\alpha_a\phi(x_a)} \right\rangle_0 &= r_0^{\sum_a \alpha_a^2 + \sum_j \beta_j^2} \prod_{a<b} |z_a - z_b|^{2\alpha_a\alpha_b} \times \\ &\times \prod_{j<j'} |w_j - w_{j'}|^{2\beta_a\beta_b} \prod_{a,j} \left(\frac{w_j - z_a}{\bar{w}_j - \bar{z}_a} \right)^{\alpha_a\beta_j} \times \begin{cases} 1, & \sum \alpha_a = \sum \beta_j = 0; \\ 0 & \text{otherwise.} \end{cases} \end{aligned} \quad (25)$$

Partition function in terms of the free boson

This coincides with the integrand of $Z[J]$ if

$$\alpha_a = \sqrt{\frac{\pi}{g}} q_a, \quad \beta_j = \sqrt{\frac{g}{4\pi}} J_j. \quad (26)$$

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Moreover, the Lagrangian of the free field is written identically in terms of both the fields ϕ and $\tilde{\phi}$. Thus we can identify

$$\varphi(x) = \sqrt{\frac{g}{4\pi}} \tilde{\phi}(x). \quad (27)$$

Sine-Gordon theory

Since $r_0^{\frac{\pi}{g}q^2} \ll r_0^{q\frac{\pi}{g}}$, we may neglect the contribution of a q -vortex compared to the contribution of q instances of a 1-vortex.

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where

$$S_{\text{SG}}[\phi] = \int d^2x \left(\frac{(\partial_\mu \phi)^2}{8\pi} - \mu : \cos \beta \phi : \right) \tag{29}$$

is the action of the sine-Gordon model with the parameters

$$\beta = \sqrt{\frac{\pi}{g}}, \quad \mu = 2r_0^{\frac{\pi}{g}-2}. \tag{30}$$

Scaling dimension of the perturbation term

The sine-Gordon model is a perturbation of the free massless fermion model with the perturbation term $\sim \cos \beta \phi$ in the Lagrangian with the scaling dimension

$$\Delta_p = \beta^2 = \frac{\pi}{g}.$$

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- 2 $\Delta_p > 2$ ($g < g_{\text{BKT}}$). The perturbation is **irrelevant** and **nonrenormalizable**. It changes the infrared behavior breaking the perturbation theory beyond the leading (tree) contributions. The infrared behavior remains free-fermion-like.

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$$\Delta_p = \beta^2 = \frac{\pi}{g}.$$

There are three regimes:

- 1 $\Delta_p < 2$ ($g > g_{\text{BKT}}$). The perturbation is **relevant** and **superrenormalizable**. It does not change the ultraviolet behavior of the theory, but essentially changes the infrared behavior.
- 2 $\Delta_p > 2$ ($g < g_{\text{BKT}}$). The perturbation is **irrelevant** and **nonrenormalizable**. It changes the infrared behavior breaking the perturbation theory beyond the leading (tree) contributions. The infrared behavior remains free-fermion-like.
- 3 $\Delta_p = 2$ ($g = g_{\text{BKT}}$). The perturbation is **marginal**. In the case of the sine-Gordon theory it is also **renormalizable**. Nevertheless it changes both infrared and ultraviolet behavior.

1. Define

$$\varphi(z) = Q - iP \log z + \sum_{k \neq 0} \frac{a_k}{ik} z^{-k},$$

$$[P, Q] = -i, \quad [a_k, a_l] = k\delta_{k+l,0},$$

$$P|0\rangle = a_k|0\rangle = 0, \quad \langle 0|a_{-k} = 0 \quad (k > 0).$$

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$$e^{i\alpha\varphi(r_0, z)} = \exp \left(i\alpha Q + \alpha P \log z + \alpha \sum_{k>0} \left(\frac{a_k}{k} z^{-k} - \frac{a_{-k}}{k} (z - r_0)^k \right) \right),$$

$$:e^{i\alpha\varphi(r_0, z)}: = e^{i\alpha Q} z^{\alpha P} \exp \left(-\alpha \sum_{k>0} \frac{a_{-k}}{k} (z - r_0)^k \right) \exp \left(\alpha \sum_{k>0} \frac{a_k}{k} z^{-k} \right).$$

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Calculate the coefficient:

$$e^{i\alpha\varphi(r_0, z)} = r_0^{\alpha^2/2} :e^{i\alpha\varphi(r_0, z)}:.$$

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$$\langle :e^{i\sum_{j=1}^N \alpha_j \varphi(z_j)}: \rangle = \begin{cases} 1, & \sum_j \alpha_j = 0; \\ 0 & \text{otherwise.} \end{cases}$$

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6. Prove that this correlation function is invariant under the transformation

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