Lecture 12
Kondo problem: solving Bethe equations

In the last lecture the Bethe equations were obtained for the sd-model:

$$e^{ip_a L} = e^{iJS} \prod_{i=1}^{n} \frac{v_i + i/2}{v_i - i/2}, \tag{1}$$

$$\left(\frac{v_i + i/2}{v_i - i/2}\right)^N \frac{v_i + iS + g^{-1}}{v_i - iS + g^{-1}} = -\prod_{j=1}^{n} \frac{v_i - v_j + i}{v_i - v_j - i}, \tag{2}$$

$$a = 1, \ldots, N, \qquad i, j = 1, \ldots, n,$$

while

$$g = \frac{1}{S + 1/2} \operatorname{tg} J(S + 1/2). \tag{3}$$

The energy of the system is[1]

$$E = \sum_{a=1}^{N} p_a. \tag{4}$$

Take the logarithm of the Bethe equations:

$$p_a L = 2\pi I_a + JS - \sum_{i=1}^{n} (\pi + p(v_i)), \tag{5}$$

$$N p(v_i) + \delta_S(v_i) = 2\pi J_i + \sum_{j=1}^{n} \Phi(v_i - v_j), \tag{6}$$

$$p(v) = 2 \operatorname{arctg} 2v, \qquad \delta_S(v) = p((v + g^{-1})/2S), \qquad \Phi(v) = p(v/2), \tag{7}$$

$$I_a \in \mathbb{Z}, \qquad J_i \in \mathbb{Z} + \frac{N - n}{2}. \tag{8}$$

Besides, all the numbers J_i should be pairwise distinct, and all the numbers I_a too.

The total energy of the system (4) is split into two contributions:

$$E = E_{\text{ch}} + E_{\text{sp}}, \tag{9}$$

$$E_{\text{ch}} = \frac{2\pi}{L} \sum_{a=1}^{N} I_a - \frac{\pi N^2}{2L}, \tag{10}$$

$$\begin{aligned} E_{\text{sp}} &= \frac{\pi N^2}{2L} + \frac{NJS}{L} - \frac{N}{L} \sum_{i=1}^{n} (\pi + p(v_i)) \\ &= -\frac{2\pi}{L} \sum_{i=1}^{n} J_i + \frac{\pi}{L} N \left(\frac{N}{2} - n\right) + \frac{NJS}{L} + \frac{1}{L} \sum_{i=1}^{n} \delta_S(v_i), \end{aligned} \tag{11}$$

The term $-\pi N^2/2L$ is added to the charge energy, so that for $J = 0$, $n = N/2$ the ratio $E_{\text{sp}}/E_{\text{ch}}$ vanishes in the thermodynamic limit.

First, we find the ground state. In order for the energy minimization procedure to be correctly defined, the cutoff $-\epsilon_F$ for negative momenta should be introduced. Namely, it should be required that in the N-particle ground state all levels of negative momentum (energy) in the interval $[-\epsilon_F, 0]$ should be filled. Since the density of states in p_a is equal to $2\frac{L}{2\pi}$, we have

$$N = \frac{L\epsilon_F}{\pi}. \tag{12}$$

[1] We have assumed $v_F = 1$. In order to restore the physical definitions of variables, one should everywhere replace $E \to E/v_F$, $J \to J/v_F$.

Therefore, the thermodynamic limit is defined as

$$L \to \infty, \qquad N \to \infty, \qquad \frac{N}{L} = \frac{\epsilon_F}{\pi} = \text{const}\,,$$

The Fermi level ϵ_F should be considered as a model parameter.

The numbers I_a must satisfy the condition

$$I_a \gtrsim -\frac{\epsilon_F L}{2\pi} = -\frac{N}{2}.$$

In the ground state I_a run through values from about $-N/2$ to $N/2$ and, therefore, $\sum_a I_a \ll N^2$. Therefore, in the thermodynamic limit the charge energy is

$$E_{\text{ch}} = -\frac{\pi N^2}{2L} = -L\frac{{\epsilon_F}^2}{2\pi}. \tag{13}$$

Find a allowed domain for the numbers J_i. For $v_i \to +\infty$ from (6) we have $J_i \to (N+2-n)/2$, and for $v_i \to -\infty$ we have $J_i \to -(N+2-n)/2$. Therefore

$$-\frac{N+1-n}{2} \leq J_i \leq \frac{N+1-n}{2}. \tag{14}$$

To find the minimum by the spin states of the electrons, suppose that the constant J is small enough, so that the last two terms in (11) can be neglected. The first term in E_{sp} decreases with increasing J_i, so we can assume that the ground state corresponds to the full filling of (half)integers by sufficiently large values of J_i. Therefore, there is some value of $J_{\min}$ such that in the ground state the roots correspond to all

$$J_{\min} \leq J_i \leq \frac{N+1-n}{2}. \tag{15}$$

In the variables v_i this corresponds to the interval

$$-b \leq v < +\infty \tag{16}$$

Assuming

$$J_i = \frac{N-n}{2} + 1 - i,$$

we obtain

$$J_{\min} = \frac{N-3n}{2} + 1 \tag{17}$$

and

$$\sum_{i=1}^{n} J_i = n\left(\frac{N+1}{2} - n\right). \tag{18}$$

Note that from (17) and (15) it follows that

$$n \leq \frac{N+1}{2} \qquad \Leftrightarrow \qquad S^z \geq S - \frac{1}{2}, \tag{19}$$

where $S^z = N/2 + S - n$ is the projection of the total spin of the system.

Let us calculate the magnetization and spin energy of the system in the leading order in N^{-1}. In this approximation we may neglect the contribution of the impurity, and we obtain

$$M \equiv \frac{S^z}{N} \simeq \frac{1}{2} - \frac{n}{N}, \tag{20}$$

and

$$E_{\text{sp}} \approx 2\epsilon_F \frac{(S^z)^2}{N} = 2N\epsilon_F M^2. \tag{21}$$

From this it is easy to obtain a relation between S^z and the magnetic field H. Indeed, by minimizing the function $E_{\rm sp}^{\rm el}(H) = E_{\rm sp}^{\rm el} - HS^z$ with respect to S^z we find

$$H = \frac{4\epsilon_F}{N} S^z. \tag{22}$$

This is simply the contribution of s-electrons to the Pauli paramagnetism. This formula is exact in the zeroth order in $1/N$ and can be used later to calculate the relationship between H and b. To obtain (22) we did not need to explicitly solve the Bethe equations (29). However, they will certainly have to be solved if we want to establish a connection between b and n.

By taking the difference of two equations (6) with consequent values of i and dividing it by $v_{i-1} - v_i$, in the thermodynamic limit we obtain the integral equation

$$\rho(v) = a_1(v) + \frac{1}{N} a_{2S}(v + g^{-1}) - \int_{-b}^{\infty} \frac{dv'}{2\pi}\, a_2(v - v')\rho(v'), \qquad -b \le v < \infty. \tag{23}$$

Here $\rho(v) = \frac{2\pi}{N}\frac{dJ}{dv}$, and

$$a_t(v) = \frac{t}{v^2 + t^2/4}. \tag{24}$$

Wherein

$$n = N \int_{-b}^{\infty} \frac{dv}{2\pi}\, \rho(v). \tag{25}$$

This means that the total spin is

$$S^z = \frac{N}{2} + S - N \int_{-b}^{\infty} \frac{dv}{2\pi}\, \rho(v). \tag{26}$$

The spin energy is equal to

$$E_{\rm sp} = \frac{N\epsilon_F}{2} + \frac{\epsilon_F JS}{\pi} - \frac{N\epsilon_F}{\pi} \int_{-b}^{\infty} \frac{dv}{2\pi}\, \rho(v)(\pi + p(v)). \tag{27}$$

Expand the density in powers of $1/N$ up to the first order

$$\rho(v) = \rho_0(v) + \frac{\rho_1(v)}{N}. \tag{28}$$

The equation for ρ_0

$$\rho_0(v) = a_1(v) - \int_{-b}^{\infty} \frac{dv'}{2\pi}\, a_2(v - v')\rho_0(v'), \qquad -b \le v < \infty, \tag{29}$$

coincides with the integral equation for the XXX model. By subtracting (29) from (23) we obtain

$$\rho_1(v) = a_{2S}(v + g^{-1}) - \int_{-b}^{\infty} \frac{dv'}{2\pi}\, a_2(v - v')\rho_1(v'), \qquad -b \le v < \infty. \tag{30}$$

The magnetization splits into the electronic and the impurity parts:

$$S^z = NM_{\rm el} + M_{\rm im}, \qquad M_{\rm el} = \frac{1}{2} - \int_{-b}^{\infty} \frac{dv}{2\pi}\, \rho_0(v), \qquad M_{\rm im} = S - \int_{-b}^{\infty} \frac{dv}{2\pi}\, \rho_1(v). \tag{31}$$

The spin energy splits into two parts

$$E_{\rm sp} = E_{\rm sp}^{\rm el} + E_{\rm im}, \tag{32}$$

$$E_{\rm sp}^{\rm el} = \epsilon_F \left(\frac{N}{2} - n\right) - 2\epsilon_F \int_{-b}^{\infty} \frac{dv}{2\pi}\, J_0(v)\rho_0(v), \tag{33}$$

$$E_{\rm im} = \frac{\epsilon_F JS}{\pi} - \frac{\epsilon_F}{\pi} \int_{-b}^{\infty} \frac{dv}{2\pi}\, \rho_1(v)(\pi + p(v)). \tag{34}$$

The electronic part of the energy is not difficult to calculate without solving the integral equation and it coincides with the result (21). To take into account the impurity we have to solve the integral equations.

Let us start with the case $b = \infty$. The integral equations (29) and (30) in this case can be solved by the Fourier method. It is easy to check that

$$\tilde{a}_t(k) = \int_{-\infty}^{\infty} \frac{dv}{2\pi}\, a_t(v) e^{ikv} = e^{-t|k|/2}. \tag{35}$$

From this we have

$$\tilde{\rho}_0(k) = e^{-|k|/2} - e^{-|k|}\tilde{\rho}_0(k), \qquad \tilde{\rho}_1(k) = e^{-S|k|-ik/g} - e^{-|k|}\tilde{\rho}_1(k).$$

Therefore,

$$\tilde{\rho}_0(k) = \frac{1}{2\,\mathrm{ch}\,\frac{k}{2}}, \qquad \tilde{\rho}_1(k) = \frac{e^{-(S-1/2)|k|-ik/g}}{2\,\mathrm{ch}\,\frac{k}{2}}. \tag{36}$$

The point $k = 0$ is of particular interest:

$$\tilde{\rho}_0(0) = \int \frac{dv}{2\pi}\, \rho_0(v) = \frac{1}{2}, \qquad \tilde{\rho}_1(0) = \int \frac{dv}{2\pi}\, \rho_1(v) = \frac{1}{2}. \tag{37}$$

From this we obtain

$$M_{\mathrm{el}} = 0, \tag{38a}$$
$$M_{\mathrm{im}} = S - 1/2. \tag{38b}$$

The first formula (38a) means that the case $b = -\infty$ corresponds to the case of zero electron magnetic moment, i.e. of zero external magnetic field. More precisely, this corresponds to the limit $H \to 0^+$, since a finite magnetic field corresponds to finite b. The formula (38b) means that in a weak magnetic field the chain acquires the angular momentum $S^z = S - 1/2$ in consistency with (19), that is, the spin of the chain is $S - 1/2$. This means that the impurity spin is partially screened by electrons and *the ground state is* $2S$*-fold degenerate*. Note that the solution (38) exactly corresponds to the maximal value of n defined in (19). Thus this result could be obtained without solving the integral equations.

Consider now the case

$$1 \ll b < \infty. \tag{39}$$

The condition $b \gg 1$ corresponds to the physically meaningful regime of the not too strong magnetic field $H \ll \epsilon_F$. Integral equations with one finite limit are solved by the Wiener–Hopf method. Let us briefly outline this method.

Consider the equation

$$f(x) + \int_0^{\infty} \frac{dx'}{2\pi}\, K(x - x') f(x') = g(x), \qquad x > 0. \tag{40}$$

The given function $g(x)$ can be arbitrarily continued to the negative x region and the equation can be extended to the entire axis. Moreover, the values of $f(x)$ for $x < 0$ are not significant, and the solution $f(x)$ for $x > 0$ if independent of this continuation.

Make the Fourier transform:

$$\tilde{f}_+(k) = \int_0^{\infty} \frac{dx}{2\pi}\, e^{ikx} f(x), \qquad \tilde{f}_-(k) = \int_{-\infty}^{0} \frac{dx}{2\pi}\, e^{ikx} f(x). \tag{41}$$

The function $\tilde{f}_+(k)$ ($\tilde{f}_-(k)$) has no singularities in the upper (lower) half-plane. Here and below, such a property will be assumed for all functions with the $\pm$ subscripts.

The equation (40) takes the form

$$(1 + \tilde{K}(k))\tilde{f}_+(k) + \tilde{f}_-(k) = \tilde{g}(k). \tag{42}$$

Represent the kernel $\tilde{K}(k)$ in the form

$$1 + \tilde{K}(k) = \frac{\tilde{K}_+(k)}{\tilde{K}_-(k)}. \tag{43}$$

Besides, we set

$$\tilde{K}_-(k)\tilde{g}(k) \equiv \tilde{q}(k) = \tilde{q}_+(k) + \tilde{q}_-(k). \tag{44}$$

For a "good" enough function $\tilde{q}(k)$ the functions $\tilde{q}_\pm(k)$ are explicitly found in the form

$$\tilde{q}_\pm(k) = \pm \int_{-\infty}^{\infty} \frac{dk'}{2\pi i} \frac{\tilde{q}(k')}{k' - k \mp i0}. \tag{45}$$

Multiplying (42) by $\tilde{K}_-(k)$, we obtain

$$\tilde{K}_+(k)\tilde{f}_+(k) + \tilde{K}_-(k)\tilde{f}_-(k) = \tilde{q}_+(k) + \tilde{q}_-(k). \tag{46}$$

Let us transfer all functions that do not have singularities in the upper half-plane to the left side:

$$\tilde{K}_+(k)\tilde{f}_+(k) - \tilde{q}_+(k) = \tilde{q}_-(k) - \tilde{K}_-(k)\tilde{f}_-(k).$$

The left-hand side has no singularities in the upper half-plane, and the right-hand side in the lower one. Thus, both sides of this equation have no singularities. Under some additional restrictions on the growth of the functions (which must be checked separately in each case), it follows that

$$\tilde{K}_+(k)\tilde{f}_+(k) = \tilde{q}_+(k), \qquad \tilde{K}_-(k)\tilde{f}_-(k) = \tilde{q}_-(k). \tag{47}$$

Finally,

$$f(x) = \int_{-\infty}^{\infty} dk \, \frac{\tilde{q}_+(k)}{\tilde{K}_+(k)} e^{-ikx}, \qquad x > 0. \tag{48}$$

The construction of functions analytic in the upper or lower half-plane is a kind of art, but for reasonable functions expressed in terms of elementary functions, this is a quite solvable problem (which reduces, more or less, to counting poles and zeros).

In the problem that we are considering, it is better to solve the equation lightly indirectly, since this will allow us to simplify the problem for $b \gg 1$, which corresponds to the physical condition $H \ll \epsilon_F$. Let

$$f_i(x) = \rho_i(x - b).$$

Then

$$\tilde{K}(k) = e^{-|k|}, \qquad \tilde{g}_0(k) = e^{ikb - |k|/2}, \qquad \tilde{g}_1(k) = e^{ikb - ik/g - S|k|}. \tag{49}$$

Rewrite equation (42) in the form

$$\tilde{f}_{i+}(k) + \frac{\tilde{f}_{i-}(k)}{1 + \tilde{K}(k)} = \frac{\tilde{g}_i(k)}{1 + \tilde{K}(k)}, \tag{50}$$

and then perform the inverse Fourier transform:

$$f_i(x) + \int_{-\infty}^{0} \frac{dx'}{2\pi} R(x - x') f_i(x') = h_i(x), \tag{51}$$

where

$$R(x) = \int_{-\infty}^{\infty} dk \, e^{-ikx} \left(\frac{1}{1 + \tilde{K}(k)} - 1 \right) = - \int_{-\infty}^{\infty} dk \, \frac{e^{-ikx}}{1 + e^{|k|}},$$
$$h_0(x) = \frac{\pi}{\operatorname{ch} \pi(x - b)}, \qquad h_1(x) = \int_{-\infty}^{\infty} dk \, e^{-ik(x - b + g^{-1})} \frac{e^{-(2S-1)|k|/2}}{2 \operatorname{ch} \frac{k}{2}}. \tag{52}$$

For $b \gg 1$ to calculate $f_0(x)$ for $x < 0$ we may use the approximation

$$h_0(x) \simeq 2\pi e^{\pi(x - b)}. \tag{53}$$

Thus, $\tilde{f}_{0-}(k) \sim e^{-\pi b}$. Since $\tilde{K}(0) = \tilde{g}_i(0) = 1$, we gave

$$2\tilde{f}_{i+}(0) + \tilde{f}_{i-}(0) = 1$$

From this we obtain

$$\frac{H}{4\epsilon_F} = M_{\rm el} = \frac{1}{2} - \int_{-b}^{\infty} \frac{dv}{2\pi}\,\rho_0(v) = \frac{1}{2} - \tilde{f}_{0+}(0) = \frac{1}{2}\tilde{f}_{0-}(0) \sim e^{-\pi b}.$$

The exact answer requires a careful solution of the equation by the Wiener–Hopf method and gives

$$\frac{H}{2\epsilon_F} = e^{-\pi b}\left(\frac{2}{\pi e}\right)^{1/2}. \tag{54}$$

Let us obtain one more simple result for $S = 1/2$. Consider the limit of small fields H when b is large, but still not equal to infinity. Then, in addition to (53) we have

$$h_1(x) \simeq 2\pi e^{\pi(x-b)+\pi/g}. \tag{55}$$

From this we immediately obtain

$$\frac{\tilde{f}_{1-}(k)}{\tilde{f}_{0-}(k)} = e^{\pi/g}.$$

Hence for the impurity contribution to the magnetic susceptibility we have

$$\chi_{\rm im} = \frac{M_{\rm im}}{H} = \frac{e^{\pi/g}}{4\epsilon_F}, \quad \text{if } S = 1/2. \tag{56}$$

I will not give explicit formulas for solving the equations (29), (30) by the Wiener-Hopf method (this conclusion is described in detail in [1]). I will only give the answer. Fro finite field the magnetization is given by the formula [2]

$$M_{\rm im}(H) = S - \frac{1}{2} + \frac{i}{4\pi^{3/2}}\int_{-\infty}^{\infty} d\omega \left(\frac{H}{T_H}\right)^{-2i\omega} \frac{\Gamma(i\omega+1/2)}{\omega+i0}\left(\frac{-i\omega+0}{e}\right)^{-2iS\omega}\left(\frac{i\omega+0}{e}\right)^{i(2S-1)\omega}, \tag{57}$$

where the variable ω is nothing but $k/2\pi$, and

$$T_H = \left(\frac{2\pi}{e}\right)^{1/2}\frac{2\epsilon_F}{\pi}e^{-\pi/g} \sim T_K. \tag{58}$$

This expression can be expanded for $H \gg T_H$ (which can be compared with the series of the perturbation theory) and for $H \ll T_H$ (which is unattainable by the perturbation theory). In the leading asymptotics, we have

$$M_{\rm im}(H) = S\left(1 - \frac{1}{\log(H/T_H)^2} - \frac{\log\log(H/T_H)^2}{\log^2(H/T_H)^2} + \cdots\right), \qquad H \gg T_H, \tag{59}$$

and

$$\begin{gathered} M_{\rm im}(H) = (S-1/2)\left(1 + \frac{1}{\log(T_H/H)^2} - \frac{\log\log(T_H/H)^2}{\log^2(T_H/H)^2} + \cdots\right), \qquad H \ll T_H, \quad S > 1/2; \\ M_{\rm im}(H) = \sqrt{\frac{2}{\pi e}}\frac{H}{T_H} + \cdots, \qquad H \ll T_H, \quad S = 1/2. \end{gathered} \tag{60}$$

Now we briefly touch on the problem of calculating the thermodynamic characteristics at finite temperatures. There are several features here.

First of all, unlike the ground state, the roots of the Bethe equations v_i can be not only real, but also complex. Namely, for $N \to \infty$ the roots of the Bethe equations form *p-strings* ($p = 1, 2, \ldots$):

$$v^p_{j,k} = v^p_j + \frac{i}{2}(p + 1 - 2k) + O(e^{-\,\mathrm{const}\,N}), \qquad k = 1, 2, \ldots, p. \tag{61}$$

Real roots correspond to 1-strings.

It can be shown that if we substitute a string solution into the Bethe equations, the right-hand side will go to zero or infinity, while the left-hand side will tend to zero or, correspondingly, to infinity as $N \to \infty$.

To construct the equations for the real parts v_j^p of strings $v_{j,k}^p$, multiply the p Bethe equations for all values of k. After that, the Bethe equations take the form

$$e^{ip_a L} = e^{iJS} \prod_{p=1}^{\infty} \prod_{j=1}^{n_p} e_p(v_j^p), \tag{62}$$

$$(e_p(v_j^p))^N e_{p,2S}(v_j^p + g^{-1}) = \prod_{p'=1}^{\infty} \prod_{j'=1}^{n_m} E_{pp'}(v_j^p - v_{j'}^{p'}), \tag{63}$$

where

$$e_p(v) = -e^{iP_p(v)} = \frac{v + ip/2}{v - ip/2}, \qquad e_{p,S}(v) = -e^{i\Delta_{p,S}(v)} = \prod_{k=1}^{p} \frac{v + \frac{i}{2}(p+1-2k) + iS}{v + \frac{i}{2}(p+1-2k) - iS},$$
$$E_{pp'}(v) = e^{i\Phi_{pp'}(v)} = e_{|p-p'|}(v) e_{|p-p'|+2}^2(v) \dots e_{p+p'-2}^2(v) e_{p+p'}(v). \tag{64}$$

Evidently,

$$S^z = \frac{N}{2} - \sum_{p=1}^{\infty} p n_p.$$

As before, we should take logarithm of the equations and pass to the continuous limit. However, at a nonzero temperature, the states are not filled densely, therefore, if the *density of states* $\rho_p(v)$ appears on the left-hand side of the integral equations, then only the *density of particles* $\rho_p^\bullet(v)$ appears in the right-hand side (under the integrals). The difference $\rho_p^\circ(v) = \rho_p(v) - \rho_p^\bullet(v)$ represents the *density of holes*. The densities of states are expressed in terms of the densities of particles according to the equation

$$\rho_p(v) = P_p(v) + \frac{1}{N}\Delta_{p,S}(v) + \sum_{p'} \int \frac{dv'}{2\pi} \Phi_{pp'}(v - v') \rho_{p'}^\bullet(v'). \tag{65}$$

Since the densities do not define the states uniquely, the entropy is associated with them:

$$S = \log \prod_{p,v} \frac{(N\rho_p(v)\frac{dv}{2\pi})!}{(N\rho_p^\bullet(v)\frac{dv}{2\pi})!(N\rho_p^\circ(v)\frac{dv}{2\pi})!}$$
$$= N \sum_{p=1}^{\infty} \int \frac{dv}{2\pi} \left(\rho_p(v) \log \rho_p(v) - \rho_p^\bullet(v) \log \rho_p^\bullet(v) - \rho_p^\circ(v) \log \rho_p^\circ(v)\right). \tag{66}$$

The correct system of equations for the density of particles is the minimum condition of the free energy

$$F[\rho^\bullet] = E - TS - HS^z$$

This gives a system of nonlinear integral *Yang–Yang equations*:

$$\epsilon_p(v) + \sum_{p'} \int \frac{dv'}{2\pi} \Phi_{pp'}(v - v') \log(1 + e^{-\epsilon_{p'}(v')}) = T^{-1} \left(P_p(v) + \frac{1}{N}\Delta_{p,S}(v) + pH \right), \tag{67}$$

where the *pseudoenergies* $\epsilon_p(v)$ are defined by the relation

$$\frac{\rho_p^\bullet(v)}{\rho_p(v)} = \frac{1}{e^{\epsilon_p(v)} + 1}.$$

These equations can be solved analytically in the limit of low or high temperatures or numerically and finite temperatures. Thermodynamic quantities are expressed in terms of pseudoenergies.

Bibliography

[1] N. Andrei, K. Furuya, and J. H. Lowenstein, *Reviews of Modern Physics*, **55** (1983) 331.

[2] V. Fateev, P. Wiegmann, *Phys. Lett. A* **81** (1981) 179.

Problems

1. Derive the expression (21) from the expression (33).

2. Show that $e^{-\pi|\omega|} = \left(\frac{i\omega+0}{e}\right)^{i\omega}\left(\frac{-i\omega+0}{e}\right)^{-i\omega}$. Solve the equation (51) for $i = 0$ in the approximation (53) by the Wiener–/Hopf method and derive (54). Check the convergence of the integrals.

3. Solve the equations (30) by the Wiener–/Hopf method and derive the formula (57) for the impurity magnetization. Check the convergence of the integrals.

4. Obtain the first nontrivial terms in the expansions (59), (60).

5*. Obtain the Bethe equations (62)–(64) for "string" solutions.