**Lecture 10**
**Algebraic Bethe Ansatz. Solving Bethe equations**

Let us return to the definition of the $L$-operator:

$$L(u) = R_{0N}(u) \dots R_{02}(u) R_{01}(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}. \tag{1}$$ Ludef

Consider the matrix element $B(u) = L(u)^+_-$. This element decreases spin by one:

$$[S^z, B(u)] = -B(u). \tag{2}$$ SzBcom

Apply this operator to the pseudo-vacuum $|\Omega_+\rangle$. We will obtain a plane wave

$$B(u)|\Omega_+\rangle = \sum_j b^{j-1}(u) c(u) a^{N-j}(u) |j\rangle = \frac{a^N(u)c(u)}{b(u)} \sum_j \left(\frac{b(u)}{a(u)}\right)^j |j\rangle. \tag{3}$$ B1stat

The role of momentum is played by the parameter $u$, which determines the ratio $z = b(u)/a(u)$. A bit more difficult to obtain the following state:

$$B(u_1)B(u_2)|\Omega_+\rangle = \frac{a_1^N a_2^N c_1 c_2}{b_1 b_2} \sum_{j_1<j_2} \left(\frac{a_{21}}{b_{21}} z_1^{j_1} z_2^{j_2} + \frac{a_{12}}{b_{12}} z_1^{j_2} z_2^{j_1}\right) |j_1, j_2\rangle, \tag{4}$$ B2stat

where $a_i = a(u_i)$, $a_{ij} = a(u_i - u_j)$ and so on, $z_i = b_i/a_i$. It can be checked by a straightforward calculation that

$$S(z_1, z_2) = \frac{a(u_1 - u_2)b(u_2 - u_1)}{b(u_1 - u_2)a(u_2 - u_1)}, \qquad z_i = \frac{b(u_i)}{a(u_i)}. \tag{5}$$ Sz1z2-

On the basis of these examples we may suppose that the states of the form

$$|u_1, u_2, \dots, u_k\rangle = B(u_1)B(u_2)\dots B(u_k)|\Omega_+\rangle \tag{6}$$ Bstate

have the structure of the Bethe wave functions with $z_j = b(u_j)/a(u_j)$. The expression (6) is called the *algebraic Bethe Ansatz.* To understand whether this expression really gives eigenvectors, consider the commutation relations that follow from the Young–Baxter equation:

$$R_{12}(u_1 - u_2)L_1(u_1)L_2(u_2) = L_2(u_2)L_1(u_1)R_{12}(u_1 - u_2).$$

First, the ${}^{++}_{--}$-component of this relation gives

$$B(u_1)B(u_2) = B(u_2)B(u_1). \tag{7}$$ BBcomm

This means that the states (6) are symmetric in $u_1, \dots, u_k$. Second, from the components ${}^{++}_{-+}$ and ${}^{-+}_{--}$ we have the relations

$$a(u_1 - u_2)B(u_1)A(u_2) = c(u_1 - u_2)B(u_2)A(u_1) + b(u_1 - u_2)A(u_2)B(u_1), \tag{8}$$ ABcomm

$$a(u_2 - u_1)B(u_1)D(u_2) = c(u_2 - u_1)B(u_2)D(u_1) + b(u_2 - u_1)D(u_2)B(u_1). \tag{9}$$ DBcomm

From these relations one can derive the identities

$$\begin{aligned} A(u)|u_1, \dots, u_n\rangle &= \alpha(u; u_1, \dots, u_n)|u_1, \dots, u_n\rangle \\ &\quad - \sum_{i=1}^k \frac{c(u_i - u)}{b(u_i - u)} \alpha(u_i; u_1, \dots, \widehat{u_i}, \dots, u_n)|u, u_1, \dots, \widehat{u_i}, \dots, u_n\rangle, \\ D(u)|u_1, \dots, u_n\rangle &= \delta(u; u_1, \dots, u_n)|u_1, \dots, u_n\rangle \\ &\quad - \sum_{i=1}^n \frac{c(u - u_i)}{b(u - u_i)} \delta(u_i; u_1, \dots, \widehat{u_i}, \dots, u_n)|u, u_1, \dots, \widehat{u_i}, \dots, u_n\rangle, \end{aligned} \tag{10}$$ ABacti

where

$$\alpha(u;u_1,\dots,u_n)=a^N(u)\prod_{i=1}^{n}\frac{a(u_i-u)}{b(u_i-u)},\qquad \delta(u;u_1,\dots,u_n)=b^N(u)\prod_{i=1}^{n}\frac{a(u-u_i)}{b(u-u_i)}. \tag{11}$$

The relations (10) are proved by induction.

From the relations (10) we obtain

$$T(u)|u_1,\dots,u_n\rangle=(\alpha(u;u_1,\dots,u_n)+\delta(u;u_1,\dots,u_n))|u_1,\dots,u_n\rangle+\text{bad terms}.$$

For the vector $|u_1,\dots,u_n\rangle$ to be an eigenvector, the sum of the bad terms must be zero. In this case the eigenvalue of the transfer matrix will be equal to

$$\Lambda(u;u_1,\dots,u_n)=a^N(u)\prod_{i=1}^{n}\frac{a(u_i-u)}{b(u_i-u)}+b^N(u)\prod_{i=1}^{n}\frac{a(u-u_i)}{b(u-u_i)}. \tag{12}$$

Note that the bad terms, in fact, accumulate at the points $j=1,N$ and the condition for their cancellation is equivalent to the periodicity condition.

Since $\frac{c(u)}{b(u)}=-\frac{c(-u)}{b(-u)}$, the bad terms are canceled if

$$\alpha(u_i;u_1,\dots,\widehat{u}_i,\dots,u_k)=\delta(u_i;u_1,\dots,\widehat{u}_i,\dots,u_k)$$

or

$$\left(\frac{b(u_i)}{a(u_i)}\right)^N=\prod_{\substack{j=1\\ j\neq i}}^{n}\frac{a(u_j-u_i)b(u_i-u_j)}{b(u_j-u_i)a(u_i-u_j)}, \tag{13}$$

These are the Bethe equations. Each solution of the Bethe equations corresponds to a certain eigenvector of the transfer matrix (and of the Hamiltonian), so that the states can be enumerated by the sets $\{u_i\}_{i=1}^n$.

Let us rewrite the Bethe equations more explicitly in the form

$$\left(\frac{\sin u_i}{\sin(\lambda-u_i)}\right)^N=\prod_{\substack{j=1\\ j\neq i}}^{n}\frac{\sin(u_i-u_j+\lambda)}{\sin(u_i-u_j-\lambda)}\qquad \text{for } c<a+b \text{ and so on } (|\Delta|<1), \tag{14}$$

$$\left(\frac{\operatorname{sh} u_i}{\operatorname{sh}(\lambda-u_i)}\right)^N=\prod_{\substack{j=1\\ j\neq i}}^{n}\frac{\operatorname{sh}(u_i-u_j+\lambda)}{\operatorname{sh}(u_i-u_j-\lambda)}\qquad \text{for } c>a+b\ (\Delta<-1). \tag{15}$$

The $\Delta<-1$ regime corresponds to the presence of two ground configurations of the six-vertex model, for which the configurations around all vertices are of the $c$-type, and two-fold (in the thermodynamic limit) degeneracy of the ground state of the XXZ model. Excited states in XXZ models are separated from the ground states by a gap. In this case, one speaks of the antiferroelectric ordering of the six-vertex model and the antiferromagnetic ground state of the XXZ model. In the $|\Delta|<1$ regime there are infinitely many (on an infinite lattice) ground configurations of the six-vertex model (disordered critical state) and a gapless spectrum near the ground state in the XXZ model. In both cases, the ground state corresponds to states with $S^z=0$ or $S^z=\pm\frac{1}{2}$ depending on the evenness of $n$.

Let us show how to find the largest eigenvalue $\Lambda_{\max}(u)$ in this model in the thermodynamic limit. Make the following assumptions:

1) In the ground state the plane waves contain neither exponentially growing nor exponentially decaying terms, so that $|z_i|\equiv|b(u_i)/a(u_i)|=1$ or $u_i=\lambda/2+iv_i$ with real $v_i$.

2) In the ground state, $v_i$ become dense in the thermodynamic limit, forming continuous bands without holes and separate values.

3) In the ground state $S^z/N\to 0$.

For definiteness, we will consider the case $|\Delta|<1$.

It is convenient to take logarithm of the Bethe equations. Let us introduce the notation

$$e^{ip(v)}=\frac{\sin(\lambda/2+iv)}{\sin(\lambda/2-iv)},\qquad e^{i\theta(v)}=\frac{\sin(\lambda+iv)}{\sin(\lambda-iv)}. \tag{16}$$

We choose a branch of the logarithm such that $p(0)=\theta(0)=0$. The Bethe equations are written down as

$$e^{iNp(v_i)}=(-)^{n-1}\prod_{j=1}^{n}e^{i\theta(v_i-v_j)}.$$

By taking logarithm, we obtain

$$Np(v_i)=2\pi I_i+\sum_{j=1}^{n}\theta(v_i-v_j),$$

where $I_i$ is an integer or a half integer depending on the evenness of $n$. Condition 2) can be made more exact now:

2') In the ground state, all $I_i$ form a set of consecutive integers for odd $n$ and half-integers for even $n$.

In this form, the statement seems to be true not only in the thermodynamic limit. For small $n$ it can also be shown that

2a) The largest eigenvalue of the transfer matrix in a sector with a given $S^z$ is achieved with a symmetric distribution of $I_i$ (and $v_i$) around zero.

Find the interval in which $v$ will change in the continuous limit. For this, it is more convenient to use the Hamiltonian $H_{XXZ}$. As we have already said in the last lecture, the energy of a state is the sum of the pseudoparticle energies $\epsilon(z)=2\Delta-z-z^{-1}=2\Delta-2\cos p(v)$. The lowest energy corresponds to the value $p=0$, that is, $v=0$. Pseudoparticles should densely fill the region $-p_F\le p(v)\le p_F$, where $p_F$ is the Fermi momentum, $\epsilon(e^{\pm ip_F})=\epsilon_F$. Since the function $p(v)$ is odd and monotone, the spectral parameter $v$ should run the region from $-v_F$ to $v_F$, where $p(v_F)=p_F$.

We have for the ground state

$$p(v_{i+1})-p(v_i)=\frac{2\pi}{N}+\frac{1}{N}\sum_{j=1}^{n}(\theta(v_{i+1}-v_j)-\theta(v_i-v_j)).$$

In the limit $N\to\infty$ we have

$$p'(v)=\rho(v)+\int_{-v_F}^{v_F}\frac{dv'}{2\pi}\,\theta'(v-v')\rho(v') \tag{17}$$

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or

$$\rho(v)=\frac{2\sin\lambda}{\operatorname{ch}2v-\cos\lambda}-\int_{-v_F}^{v_F}\frac{dv'}{2\pi}\,\frac{2\sin2\lambda}{\operatorname{ch}2(v-v')-\cos2\lambda}\,\rho(v'), \tag{17$'$}$$

where $p'(v)$, $\theta'(v)$ are derivatives of $p(v)$, $\theta(v)$ with respect to $v$, and $\rho(v)=\lim_{N\to\infty}\frac{1}{N(v_{i+1}-v_i)}$ is the density of roots near the point $v$, which will be an unknown function in this equation. The interval $[-v_F,v_F]$ is determined from the condition

$$\int_{-v_F}^{v_F}\frac{dv}{2\pi}\,\rho(v)=\frac{n}{N}. \tag{18}$$

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Next, we need to minimize the energy in $n/N$.

The equation (17) is easily solved by the Fourier method for $v_F=\infty$. Let

$$\rho(v)=\int dk\,\rho_k e^{ikv},\qquad p'(v)=\int dk\,p'_k e^{ikv},\qquad \theta'(v)=\int dk\,\theta'_k e^{ikv}. \tag{19}$$

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Then

$$\rho_k=p'_k-\theta'_k\rho_k,$$

i.e.

$$\rho_k=\frac{p'_k}{1+\theta'_k}.$$

It is not difficult to check that

$$p'_k=\frac{\operatorname{sh}\frac12(\pi-\lambda)k}{\operatorname{sh}\frac12\pi k},\qquad \theta'_k=\frac{\operatorname{sh}\frac12(\pi-2\lambda)k}{\operatorname{sh}\frac12\pi k}. \tag{20}$$

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From this we obtain

$$\rho_k = \frac{1}{2\operatorname{ch}\frac{1}{2}\lambda k}. \tag{21}$$

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Evidently,

$$\int_{-\infty}^{\infty} \frac{dv}{2\pi}\,\rho(v) = \rho_0 = \frac{1}{2},$$

and, therefore, $n = N/2$ and $S^z/N \ll 1$. Thus, this solution corresponds to the ground state of the system.

In the limit $N \to \infty$ one could expect that one of the terms in (12) is much larger than the other, therefore the (minus) free energy per vertex reads

$$\begin{aligned}\log\kappa(u) \equiv \lim_{N\to\infty}\frac{\log\Lambda(u)}{N} = \max\Bigg(&\log a(u) + \int_{-v_F}^{v_F}\frac{dv}{2\pi}\,\rho(v)\log\frac{a(iv-u+\lambda/2)}{b(iv-u+\lambda/2)},\\ &\log b(u) + \int_{-v_F}^{v_F}\frac{dv}{2\pi}\,\rho(v)\log\frac{a(u-iv-\lambda/2)}{b(u-iv-\lambda/2)}\Bigg).\end{aligned} \tag{22}$$

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The logarithms in the r.h.s. are easily expressed in terms of $p(v)$:

$$\begin{aligned}\log\kappa(u) = \max\Bigg(&\log a(u) - i\int_{-v_F}^{v_F}\frac{dv}{2\pi}\,\rho(v)p(iu+v),\\ &\log b(u) - i\int_{-v_F}^{v_F}\frac{dv}{2\pi}\,\rho(v)p(i(\lambda-u)+v)\Bigg).\end{aligned} \tag{23}$$

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In terms of Fourier transforms it looks like

$$\log\kappa(u) = \max\left(\log a(u) + \int\frac{dk}{k}\rho_{-k}p'_k e^{ku}, \log b(u) + \int\frac{dk}{k}\rho_k p'_k e^{k(\lambda-u)}\right). \tag{24}$$

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By substituting (20) and (21), we find that both values under the maximum sign in (22) coincide and the free energy is given by

$$\begin{aligned}\log\kappa(u) &= \log a(u) + \int_0^{\infty} dk\,\frac{\operatorname{sh}uk\operatorname{sh}\frac{\pi-\lambda}{2}k}{2k\operatorname{sh}\frac{\pi}{2}k\operatorname{ch}\frac{\lambda}{2}k}\\ &= \log b(u) + \int_{-\infty}^{\infty} dk\,\frac{\operatorname{sh}(\lambda-u)k\operatorname{sh}\frac{\pi-\lambda}{2}k}{2k\operatorname{sh}\frac{\pi}{2}k\operatorname{ch}\frac{\lambda}{2}k}.\end{aligned} \tag{25}$$

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In the case of $\Delta < -1$, the sines are replaced by hyperbolic sines and the functions $p'(v)$ and $\theta'(v)$ turn out to be periodic in $v$ with the period $\pi$. This means that the Fourier integral in (19) is replaced by the Fourier series:

$$\int\frac{dk}{2\pi} \to \frac{1}{\pi}\sum_{k\in 2\mathbf{Z}}.$$

The Fourier components are

$$p'_k = 2\pi e^{-\lambda|k|/2}, \qquad \theta'_k = 2\pi e^{-\lambda|k|}. \tag{26}$$

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The density is again given by (21), but with even integer $k$. Therefore, the final formula for the free energy has the form of a series:

$$\begin{aligned}\log\kappa(u) &= \log a(u) + u + \sum_{m=1}^{\infty}\frac{e^{-\lambda m}\operatorname{sh}2um}{m\operatorname{ch}\lambda m}\\ &= \log b(u) + \lambda - u + \sum_{m=1}^{\infty}\frac{e^{-\lambda m}\operatorname{sh}2(\lambda-u)m}{m\operatorname{ch}\lambda m}.\end{aligned} \tag{27}$$

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Note that for the general values of $v_F < \infty$ (for $|\Delta| < 1$) or $v_F < \frac{\pi}{2}$ (for $\Delta < -1$) the equation (17) describes the ground state of a six-vertex model of the general form with, generally speaking, different $a, a', b, b'$ (a nonunit ratio of $c/c'$ does not affect the model at all). The equation has no analytical solution, but can be solved numerically. In this case, two terms under the maximum sign in the free energy will be different.

## Problems

**1.** Derive (4) and (5).

**2.** Prove the relations (10) by induction in $k$.

**3.** Show that the Bethe Ansatz (13) can be obtained from the formula (12) for the eigenvalues and the requirement that for each eigenvalue $\Lambda(u)$ the product $\Lambda(u)\sin^N(\lambda - u)$ (for $|\Delta| < 1$) or $\Lambda(u)\,\mathrm{sh}^N(\lambda - u)$ (for $\Delta < -1$) be an entire function of $u$.

**4.** Introduce the $t$ variable, which will have the meaning of $(T - T_c)/T_c$ near the critical point $\Delta = -1$. Show that near this point the free energy is regular in the antiferroelectric region and has a weak singularity in the disordered region:

$$f_{\text{sing}} \sim e^{-c/(-t)^{1/2}}$$

with a certain constant $c$.

**5*.** In the asymmetric six-vertex model with $E_v = 0$ in the region $\Delta < 1$ find the value of $E_h$ that provides saturation of the electric polarization: $n = 0$. Find the corresponding magnetic field of the saturation in the $XXZ$ model.[1]

[1] For $\Delta < -1$ there is one more critical field, which bounds the antiferroelectric (or antiferromagnetic) region. But the calculation of this bound is more complicated and demands the calculation of the energy of one-particle excitations.