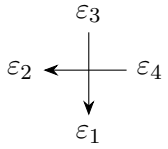
Lecture 9
Ice model and commuting transfer matrices

Consider another model of the classical statistical mechanics — the *six-vertex model* or the *ice model*. Let "spins" $\varepsilon = \pm$ live on the edges of the square lattice, while the interaction takes place at the vertices. Namely, to each configuration of spins around the top



let us put in correspondence a Boltzmann weight $R^{\varepsilon_3\varepsilon_4}_{\varepsilon_1\varepsilon_2}$. Here the arrows indicate the orientation of the lattice.

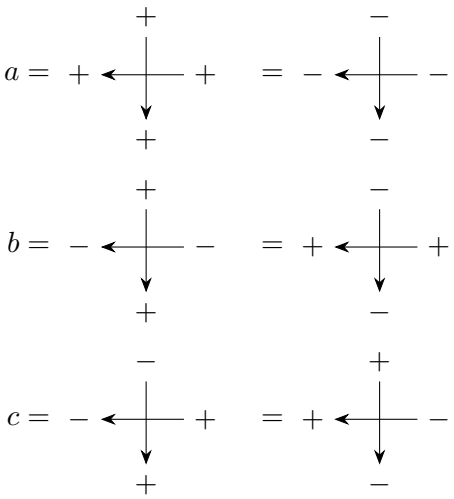
A *configuration* C is the set of values of spins on all edges of the lattice. The *weight* of a configuration $W(C)$ is the product of the Boltzmann weights at all vertices of the lattice. The *ground configuration* is the configuration of the greatest weight. The *partition function* is the sum of the weights over all configurations $Z = \sum_C W(C)$. The *six-vertex model* is the model, in which weights are not equal to zero only under the condition

$$\varepsilon_1 + \varepsilon_2 = \varepsilon_3 + \varepsilon_4, \tag{1}$$

and the weights are invariant with respect to the inversion of all spins:

$$R^{-\varepsilon_3\ -\varepsilon_4}_{-\varepsilon_1\ -\varepsilon_2} = R^{\varepsilon_3\varepsilon_4}_{\varepsilon_1\varepsilon_2}. \tag{2}$$

So, around each vertex any of the following six configurations is allowed



In this case, the matrix R can be written as

$$R = \begin{pmatrix} a & & & \\ & b & c & \\ & c & b & \\ & & & a \end{pmatrix}. \tag{3}$$

Consider a lattice model of size $M \times N$ with the cyclic boundary conditions and introduce a column transfer matrix

$$T^{\varepsilon'_1\ldots\varepsilon'_N}_{\varepsilon_1\ldots\varepsilon_N} = \sum_{\mu_1\ldots\mu_N} R^{\mu_2\varepsilon'_1}_{\mu_1\varepsilon_1} R^{\mu_3\varepsilon'_2}_{\mu_2\varepsilon_2} \ldots R^{\mu_1\varepsilon'_N}_{\mu_N\varepsilon_N}. \tag{4}$$

It is convenient to consider the matrix R as an operator on the tensor product of two two-dimensional spaces:

$$R : \mathbf{C}^2 \otimes \mathbf{C}^2 \to \mathbf{C}^2 \otimes \mathbf{C}^2, \qquad v_{\varepsilon_1} \otimes v_{\varepsilon_2} \mapsto R^{\varepsilon_1\varepsilon_2}_{\varepsilon'_1\varepsilon'_2} v_{\varepsilon'_1} \otimes v_{\varepsilon'_2}.$$

Here v_ε is the natural basis in the space $V = \mathbf{C}^2$. If there is a product of identical spaces, say, $V_1 \otimes V_2 \otimes \ldots \otimes V_k$, then by R_{ij} we will denote the operator acting on the product $V_i \otimes V_j$ as R, and on all other V_l as the unit operator. Then the transfer matrix can be written compactly in the form

$$T = \mathrm{tr}_{V_0}(R_{0N} \ldots R_{02} R_{01}). \tag{5}$$

The operator under the trace sign deserves a separate notation

$$L = R_{0N} \ldots R_{02} R_{01} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \tag{6}$$

and is called the *monodromy operator*. Evidently

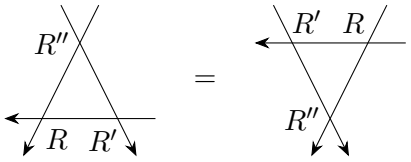
$$T = A + D. \tag{7}$$

The space $V_1 \otimes \ldots \otimes V_N$ is called the *quantum space*, and the space V_0 is called the *auxiliary space*. The operator L is usually considered as an operator in a quantum space and as a matrix in an auxiliary space. The matrix elements $A, \ldots, D$ act as operators in the quantum space.

The solution to the problem of calculating the partition function of the six-vertex model reduces to the problem of finding the eigenvalues of the transfer matrix. When could this type of problem be solvable for sure? This is essentially a question of the *quantum integrability* of a model. We do not know exactly what quantum integrability is, but in classical mechanics a model is integrable when it has a sufficient number of integrals of motion in involution. Therefore, we would like to have a sufficient number of operators commuting with the transfer matrix and with each other. Suppose that such integrals again have the form of a transfer matrix T' with some other matrix R' of the form (3). So, let there be operators T and T' of the form (5) with the R-matrices of the form (3). When do they commute? A sufficient (although not necessary) condition can be formulated as follows. The operators T and T' are commuting when there is an invertible matrix R'' of the form (3) such that

$$R''_{12} R'_{13} R_{23} = R_{23} R'_{13} R''_{12}. \tag{8}$$

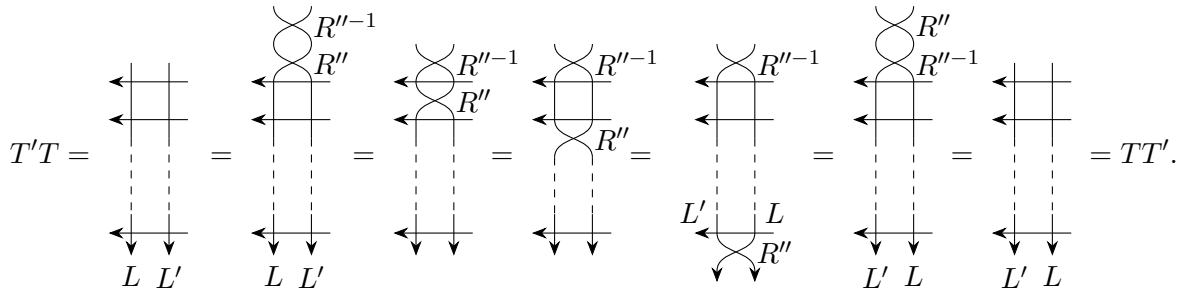
Graphically, it looks like this:



(8′)

This relation is called the *Yang–Baxter equation*.

The commutativity of the transfer matrices T and T' subject to the condition (8) is easily proved graphically (cyclic conditions on vertical lines are implied):



In the form of formulas, the proof is written as follows. From the Yang–Baxter equation it follows that

$$R''_{12} L'_1 L_2 = L_2 L'_1 R''_{12},$$

where the operators L'_1 and L_2 act on the same quantum space, but have different auxiliary spaces V_1 and V_2. Then

$$\begin{aligned} T'T &= \mathrm{tr}_{V_1 \otimes V_2}(L'_1 L_2) = \mathrm{tr}_{V_1 \otimes V_2}((R''_{12})^{-1} R''_{12} L'_1 L_2) = \mathrm{tr}_{V_1 \otimes V_2}((R''_{12})^{-1} L_2 L'_1 R''_{12}) \\ &= \mathrm{tr}_{V_1 \otimes V_2}(R''_{12}(R''_{12})^{-1} L_2 L'_1) = \mathrm{tr}_{V_1 \otimes V_2}(L_2 L'_1) = TT'. \end{aligned}$$

Now it is necessary to solve the Yang–Baxter equation. We will not derive the solution sequentially, but only give the answer. It is clear that the normalization of the R-matrices is not important, therefore the R-matrices can be parameterized by two variables. Denote them by λ and u. Since solutions of the Yang–Baxter equations are defined up to a common factor, assume $a(\lambda, u) = 1$. It is convenient to introduce the trigonometric parameterization

$$\begin{aligned} b(\lambda, u) &= \frac{\sin u}{\sin(\lambda - u)}, \\ c(\lambda, u) &= \frac{\sin \lambda}{\sin(\lambda - u)}, \end{aligned} \tag{9}$$

if $c < a + b$, $a < b + c$, $b < a + c$, and

$$\begin{aligned} b(\lambda, u) &= \frac{\operatorname{sh} u}{\operatorname{sh}(\lambda - u)}, \\ c(\lambda, u) &= \frac{\operatorname{sh} \lambda}{\operatorname{sh}(\lambda - u)}, \end{aligned} \tag{10}$$

if $c > a + b$. The cases $a > b + c$ and $b > a + c$ are not interesting (see later).

In the parameterization (9) or (10) the solution of the Yang–Baxter equation has the form

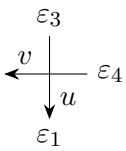
$$\begin{aligned} R &= R(\lambda, u_2 - u_3), \\ R' &= R(\lambda, u_1 - u_3), \\ R'' &= R(\lambda, u_1 - u_2). \end{aligned} \tag{11}$$

The parameter λ should be the same for all three matrices, and in the future we will omit it. The parameter u for all three matrices is different, although its values are related by a relation. It is important that for any two matrices R and R' with the same value of λ there exists a matrix R'' (with the same, again, the value λ). This means that there is a whole family of commuting transfer matrices $T(u)$ with an arbitrary u and fixed λ:

$$[T(u), T(u')] = 0 \quad \forall u, u'. \tag{12}$$

The variable u is called the *spectral parameter*.

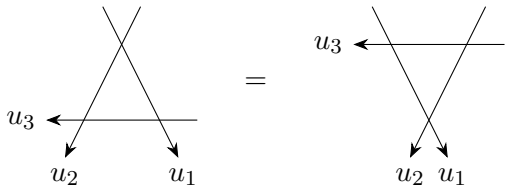
Note that the parameter u_i is conveniently assigned to the ith line, and the R-matrix can be written as

$R(u-v)^{\varepsilon_3\varepsilon_4}_{\varepsilon_1\varepsilon_2} =$ 

The Yang–Baxter relation

$$R_{12}(u_1 - u_2)R_{13}(u_1 - u_3)R_{23}(u_2 - u_3) = R_{23}(u_2 - u_3)R_{13}(u_1 - u_3)R_{12}(u_1 - u_2) \tag{13}$$

can be then depicted as

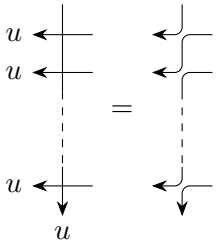


(13′)

In this form, the Yang–Baxter equation arose in field theory as a condition for the factorization of multi-particle scattering into two-particle ones. We also note that the R-matrices (9) and (10) satisfy the crossing symmetry and unitarity relations in the form

$$b(u)R(\lambda - u)^{\varepsilon_3\varepsilon_4}_{\varepsilon_1\varepsilon_2} = R(u)^{\varepsilon_2\ -\varepsilon_3}_{\varepsilon_4\ -\varepsilon_1}, \qquad R_{12}(u)R_{21}(-u) = 1, \qquad R(0) = P = \curvearrowright . \tag{14}$$

From the last equation it follows that the operator $T(0)$ is nothing but the shift operator:

$T(0) =$  (15)

The existence of a continuum family of integrals of motion does not mean, of course, that there really are infinitely many of them. In fact, some of them are dependent, so their number is finite, but sufficient for the integrability. You can get rid of the continuum parameter by taking into account the analyticity of the transfer matrix as a function of u and by considering the generating function

$$T^{-1}(0)T(u) = 1 - \sum_{n=1}^{\infty} \frac{H_n u^n}{n!}. \tag{16}$$

Hamiltonians H_n commute with $T(u)$ and mutually commute:

$$[T(u), H_n] = [H_m, H_n] = 0 \quad \forall m, n. \tag{17}$$

They form a family of local integrals of motion. Locality means that H_n can be represented as $\sum_{i=1}^{N} I_{n,i}$, where $I_{n,i}$ only depends on a finite number of nodes $i, i+1, \ldots, i+n$. In fact, only the first $N-1$ integrals H_n and the operator $T(0)$ are independent. The operator H_1 can be very easily found. Indeed, consider the product

$$\check{R}(u) = R(u)P = \begin{pmatrix} a(u) & & & \\ & c(u) & b(u) & \\ & b(u) & c(u) & \\ & & & a(u) \end{pmatrix} = 1 + \frac{u}{\sin\lambda} \begin{pmatrix} 0 & & & \\ & \cos\lambda & 1 & \\ & 1 & \cos\lambda & \\ & & & 0 \end{pmatrix} + O(u^2)$$

$$= 1 - \frac{u}{\sin\lambda}\left(h - \frac{\cos\lambda}{2}\right) + O(u^2),$$

where

$$h = -\frac{1}{2}(\sigma^x \otimes \sigma^x + \sigma^y \otimes \sigma^y - \cos\lambda\, \sigma^z \otimes \sigma^z).$$

In the first order in u it is precisely these terms that will contribute to H_1. It is not very difficult to gather them (for example, in the index notation), and obtain

$$H_1 \sin\lambda = H_{\mathrm{XXZ}} + \frac{N\Delta}{2},$$

where H_{XXZ} is the Hamiltonian of the XXZ Heisenberg model:

$$H_{\mathrm{XXZ}} = -\frac{1}{2}\sum_{n=1}^{N}(\sigma_n^x\sigma_{n+1}^x + \sigma_n^y\sigma_{n+1}^y + \Delta\sigma_n^z\sigma_{n+1}^z), \tag{18}$$

with

$$\Delta = -\cos\lambda = \frac{a^2+b^2-c^2}{2ab} \tag{19}$$

(the last equality is true for any u). In the case $c > a + b$ we have

$$\Delta = -\operatorname{ch}\lambda. \tag{19'}$$

Thus, to find the eigenstates of the six-vertex model, it is enough to find the eigenstates of the XXZ Heisenberg model. In fact, both tasks are of equal complexity. But the XXZ model contains an important hint on how to solve this problem. First of all, introduce the operator of the total spin:

$$S^z = \frac{1}{2}\sum_{n=1}^{N}\sigma_n^z.$$

It follows from the ice condition that our operators commute with it:[1]

$$[T(u), S^z] = [H_{\mathrm{XXZ}}, S^z] = 0. \tag{20}$$

It follows that the eigenstates have a certain projection of the total spin S^z. The simplest of these states are easily constructed: these are *pseudo-vacuums*

$$|\Omega_\pm\rangle = \underbrace{v_\pm \otimes v_\pm \otimes \ldots \otimes v_\pm}_{N}, \tag{21}$$

in which all spins are directed up or all down. Evidently,

$$S^z|\Omega_\pm\rangle = \pm\frac{N}{2}|\Omega_\pm\rangle, \qquad T(u)|\Omega_\pm\rangle = (a^N(u) + b^N(u))|\Omega_\pm\rangle, \qquad H_{XXZ}|\Omega_\pm\rangle = -\frac{N\Delta}{2}|\Omega_\pm\rangle.$$

The problem is that these states are ground ones only in the case of $\Delta > 1$. In this case, $a > b + c$ (or $b > a + c$) and the ground configurations of the six-vertex model are configurations, in which all spins have the same sign (or all on the vertical edges have the same sign, and on the horizontal edges the other). It is easy to see that in order to flip one spin, in this case it is also necessary to flip $\sim N$ spins, so that in the thermodynamic limit the probability of flipping the spin is exactly zero. One sais that the system has *frozen* ground configurations. We will be interested in the case $\Delta < 1$.

States of fixed spin $S^z = N/2 - k$ can be represented as linear combinations of the states

$$|n_1, \ldots, n_k\rangle = \sigma^-_{n_1} \ldots \sigma^-_{n_k}|\Omega_+\rangle, \qquad \sigma^\pm = \frac{1}{2}(\sigma^x \pm i\sigma^y), \tag{22}$$

with all n_j being different. Since the Hamiltonian H_{XXZ} only flips adjacent spins, the eigenstates of the Hamiltonian will look like plane waves when $|n_i - n_j| > 1$ $(\forall i, j)$.

We start with the case $k = 1$. In this case, we consider the state

$$|\Psi_1(z)\rangle = \sum_n z^n |n\rangle. \tag{23}$$

It is easy to verify that

$$H_{XXZ}|\Psi_1(z)\rangle = \left(-\frac{N\Delta}{2} + \epsilon(z)\right)|\Psi_1(z)\rangle, \qquad \epsilon(z) = 2\Delta - z - z^{-1}. \tag{24}$$

Evidently, $|z| = 1$, so that $-2 < z + z^{-1} < 2$. Thus, for $\Delta > 1$, all single-particle excitations have positive energy $\epsilon(z)$, so that $|\Omega_\pm\rangle$ is a true doubly degenerate vacuum of the system. In the case of $-1 < \Delta < 1$ some of the excited states are of positive energy and some are of negative energy. For $\Delta < -1$, all excitations have negative energy. In the last two cases, the ground state is a state with $S^z = 0$ or with $S^z = \pm 1/2$.

Let us now consider a two-particle state, $k = 2$. We will look for it in the form

$$|\Psi_2(z_1, z_2)\rangle = \sum_{n_1 < n_2} (A_{12} z_1^{n_1} z_2^{n_2} + A_{21} z_2^{n_1} z_1^{n_2})|n_1, n_2\rangle. \tag{25}$$

The action of the Hamiltonian on the terms of the sum with $n_2 > n_1 + 1$ does not differ from the action on the single-particle states (23), and the action on the terms with $n_2 = n_1 + 1$ can be considered as matching conditions. The later have the form

$$\frac{A_{21}}{A_{12}} = S(z_1, z_2) \equiv -\frac{1 + z_1 z_2 - 2\Delta z_2}{1 + z_1 z_2 - 2\Delta z_1}. \tag{26}$$

It is evident that

$$H_{XXZ}|\Psi_2(z_1, z_2)\rangle = \left(-\frac{N\Delta}{2} + \epsilon(z_1) + \epsilon(z_2)\right)|\Psi_2(z_1, z_2)\rangle,$$

i.e. the excitation energies sum up. In addition, it can be seen that the Ansatz for the wave function (25) does not contain reflected waves. We assume that this property is preserved in the general case:

[1] However, they *does not* commute with S^x, S^y!

Assumption. *The plane waves in the basis (22) are scattered under the action of H_{XXZ} without reflections.*

This means that wave functions should be sought in the form

$$|\Psi_k(z_1,\dots,z_k)\rangle = \sum_{n_1<\dots<n_k}\sum_{\sigma\in S_k} A_{\sigma_1\dots\sigma_k}\prod_{j=1}^{k} z_{\sigma_j}^{n_j}|n_1,\dots,n_k\rangle. \tag{27}$$

This form of wave function is called the *(coordinate) Bethe Ansatz*. It can be shown that the action of the Hamiltonian H_{XXZ} does not give rise to new matching conditions, so that the amplitude ratios are given by the same function (26):

$$A_{\dots ji\dots}/A_{\dots ij\dots} = S(z_i,z_j). \tag{28}$$

With this condition

$$H_{XXZ}|\Psi_k(z_1,\dots,z_k)\rangle = \left(-\frac{N\Delta}{2}+\sum_{i=1}^{k}\epsilon(z_i)\right)|\Psi_k(z_1,\dots,z_k)\rangle, \tag{29}$$

i.e. the excitation energies are still additive. Next, the periodicity condition should be imposed

$$z_i^N\prod_{j,\ j\neq i} S(z_i,z_j) = 1, \tag{30}$$

which gives a system of equations for the z_i "momenta" called the system of the *Bethe equations*. Solving this system and substituting solutions into (29), one can find all eigenvalues of the Hamiltonian. In the next lecture, we will obtain this system in a slightly different way.

Problems

1. Consider the asymmetric six-vertex model, i.e. the model with the R matrix

$$R = \begin{pmatrix} a_1 & & & \\ & b_1 & c_1 & \\ & c_2 & b_2 & \\ & & & a_2 \end{pmatrix}.$$

Show that in the case of the cyclic boundary conditions the model can be reduced the the symmetric model with $a=\sqrt{a_1a_2}$, $b=\sqrt{b_1b_2}$, $c=\sqrt{c_1c_2}$ in the 'horizontal' external field E_h and the 'vertical' one E_v, which produce an extra factor $e^{E_h\varepsilon}$ for each horizontal edge and $e^{E_v\varepsilon}$ for each vertical edge. Express these fields in terms of a_i,b_i. Show that these fields can be reduced to boundary conditions, though depending on the sizes of the lattice. Find the corresponding transfer matrices and demonstrate that they commute.

2. Demonstrate that in the conditions of the previous problem with $E_v=0$ the limit $u\to 0$, $E_h\to 0$ so that $E_h/u=\mathrm{const}$ provides the XXZ model in the external magnetic field. Explain, why the eigenvalues of its Hamiltonian are given by the same Bethe wave functions (27), which satisfy the same Bethe equations (30).

3. Define the "normalized" R-matrix $\tilde{R}(u)=\kappa^{-1}(u)R(u)$ by the conditions:

$$\tilde{R}(\lambda-u)^{\varepsilon_3\varepsilon_4}_{\varepsilon_1\varepsilon_2} = \tilde{R}(u)^{\varepsilon_2\ -\varepsilon_3}_{\varepsilon_4\ -\varepsilon_1}, \qquad \tilde{R}_{12}(u)\tilde{R}_{21}(-u)=1.$$

It is evident that the analytic function $\kappa(u)$ satisfies the conditions

$$\kappa(\lambda-u)b(u)=\kappa(u), \qquad \kappa(u)\kappa(-u)=1.$$

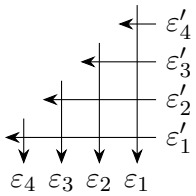
Find a solution to these equations for $\Delta<-1$, which is a function of the variable $z=e^u$ and does not have poles and zeros in the region $-\lambda<\mathrm{Re}\,u<\lambda$. It is convenient to express the solution in terms of the functions of the form

$$(z;p)_\infty = \prod_{n=0}^{\infty}(1-zp^n).$$

4. Prove the relation (26).

5*. Consider the *corner transfer matrix*, which is the partition function of a very big (nearly infinite) quadrant:

$$A(u)^{\varepsilon'_1\varepsilon'_2\ldots}_{\varepsilon_1\varepsilon_2\ldots} =$$



Each intersection contains the matrix $R(u)$. Show that $A(0) = 1$. Find the *corner Hamiltonian* $H_C = A'(u)$.