Lecture 8
Heisenberg spin chain and its scaling limit

Consider a chain of N spins $S = 1/2$, that is, the space

$$\mathcal{H}_N = \underbrace{\mathbb{C}^2 \otimes \cdots \otimes \mathbb{C}^2}_{N}, \tag{1}$$

on which the Hamiltonian acts

$$H_{XYZ} = -\frac{1}{2}\sum_{n=1}^{N}\left(J_x\sigma^x_n\sigma^x_{n+1} + J_y\sigma^y_n\sigma^y_{n+1} + J_z\sigma^z_n\sigma^z_{n+1}\right). \tag{2}$$

Here σ^i_n acts on the nth component in $\mathcal{H}_N$, and the $(N+1)$th component is identified with the first one. Such a model is called the *XYZ Heisenberg model* with cyclic boundary conditions. In the case of $J_x = J_y$ the model is called the *XXZ model*, and in the case of $J_x = J_y = \pm J_z$ the it XXX model. We will assume that N is even.

Since physics is independent of the common factor in the Hamiltonian, the following notation is usually introduced:

$$\Gamma = J_y/J_x, \qquad \Delta = J_z/J_x \tag{3}$$

It is assumed that

$$J_x > 0, \qquad |\Gamma| \le 1, \qquad |\Delta| \le |\Gamma| \quad \text{or} \quad |\Delta| \ge 1.$$

Without loss of generality, we put $J_x = 1$. The Hamiltonian is written down as

$$H_{XYZ} = -\frac{1}{2}\sum_{n=1}^{N}\left((1+\Gamma)(\sigma^+_n\sigma^-_{n+1} + \sigma^-_n\sigma^+_{n+1}) + (1-\Gamma)(\sigma^+_n\sigma^+_{n+1} + \sigma^-_n\sigma^-_{n+1}) + \Delta\sigma^z_n\sigma^z_{n+1}\right). \tag{4}$$

Here

$$\sigma^+ = \frac{\sigma^x + i\sigma^y}{2} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad \sigma^- = \frac{\sigma^x - i\sigma^y}{2} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{5}$$

are spin increasing and decreasing operators.

Let us apply the Jordan–Wigner transformation to this Hamiltonian:

$$\sigma^z_n = 2a^+_n a_n - 1, \qquad \sigma^+_n = a^+_n \exp\left(i\pi\sum_{j=1}^{n-1} a^+_j a_j\right), \qquad \sigma^-_n = a_n \exp\left(-i\pi\sum_{j=1}^{n-1} a^+_j a_j\right), \tag{6}$$

where a_n, a^+_n are fermion operators:

$$a^+_m a_n + a_n a^+_m = \delta_{mn}, \qquad a_m a_n + a_n a_m = a^+_m a^+_n + a^+_n a^+_m = 0.$$

The inverse transformation has the form

$$a_n = \sigma^-_n \prod_{j=1}^{n-1}(-\sigma^z_j), \qquad a^+_n = \sigma^+_n \prod_{j=1}^{n-1}(-\sigma^z_j). \tag{7}$$

We obtain

$$\begin{aligned} H_{XYZ} = -\sum_{n=1}^{N}\Bigg(&\frac{1+\Gamma}{2}(a^+_{n+1}a_n + a^+_n a_{n+1}) + \frac{1-\Gamma}{2}(a^+_n a^+_{n+1} - a_n a_{n+1}) \\ &+ 2\Delta(a^+_n a_n a^+_{n+1} a_{n+1} - a^+_n a_n)\Bigg). \end{aligned} \tag{8}$$

We first consider the case of *XY models*, i.e. Models with $\Delta = 0$. In this case, the Hamiltonian is quadratic in fermions and easily diagonalized. Let

$$a_n = \frac{1}{N^{1/2}}\sum_k a_k e^{ikn}. \tag{9}$$

Then

$$H_{XY} = -\sum_k \left((1+\Gamma)\cos k \cdot a_k^+ a_k + i\frac{1-\Gamma}{2}\sin k \cdot (a_k^+ a_{-k}^+ + a_k a_{-k}) \right). \tag{10}$$

Hamiltonians of this kind are diagonalized by using the Bogoliubov transform. First, consider the simple case of $\Gamma = 1$. Obviously, the Brillouin zone $-\pi \le k < \pi$ is divided into two areas: 1) $-\frac{\pi}{2} \le k < \frac{\pi}{2}$ and 2) $-\pi \le k < -\frac{\pi}{2}$ or $\frac{\pi}{2} \le k < \pi$. Evidently, the energies in the first region are negative and this part of the zone will be completely filled: $k_F = \pi/2$. In another way, we can say like this: the ground state of the system $|0\rangle$ satisfies the relation $a_k|0\rangle = 0$ in the second region and $a_k^+|0\rangle = 0$ in the first region. This dictates the following Bogoliubov transformation:

$$b_k = a_{k-\pi}, \qquad b_k^+ = a_{k-\pi}^+, \qquad b'_k = ia_k^+, \qquad b'^+_k = -ia_k, \qquad -\frac{\pi}{2} \le k < \frac{\pi}{2}, \tag{11}$$

assuming that k is defined modulo 2π. The factors $\pm i$ are introduced arbitrarily to simplify further formulas. Then the Hamiltonian will take the form

$$H_{XY} = \sum_{-\pi/2 \le k < \pi/2} 2\cos k \, (b_k^+ b_k + b'^+_k b'_k).$$

The operators b_k^+, b_k are creation-annihilation operators of the particles, while b'^+_k, b'_k of the antiparticles.

In the general case, we construct the Bogoliubov transformation so that it tended to the transformation (11) in the limit $\Gamma \to 1$. Let

$$a_k = \alpha'_k b'_k + \beta'_k b'^+_{-k}, \qquad a_k^+ = \beta'^*_k b'_{-k} + \alpha'^*_k b'^+_k, \qquad |\alpha'_k|^2 + |\beta'_k|^2 = 1, \qquad \alpha'_k \beta'_{-k} + \alpha'_{-k}\beta'_k = 0. \tag{12}$$

After substituting (12) into (10), and requiring that the terms proportional to $b'^+ b'^+$ and $b'b'$ vanish, we get

$$\alpha'_k = -\sin\frac{\kappa}{2}, \qquad \beta'_k = i\cos\frac{\kappa}{2}, \qquad \operatorname{tg}\kappa = \frac{1-\Gamma}{1+\Gamma}\operatorname{tg} k. \tag{13}$$

This solution is well defined in the first half, $-\pi/2 < k < \pi/2$, of the Brillouin zone. The remaining half will be transferred to the same area, and by replacing $\kappa \to \kappa - \pi$ we obtain

$$\alpha_k = \cos\frac{\kappa}{2}, \qquad \beta_k = i\sin\frac{\kappa}{2}. \tag{14}$$

The Hamiltonian is equal to

$$H_{XY} = \sum_{-\pi/2 < k < \pi/2} \epsilon_k (b_k^+ b_k + b'^+_k b'_k), \qquad \epsilon_k = \sqrt{(1+\Gamma)^2\cos^2 k + (1-\Gamma)^2 \sin^2 k}. \tag{15}$$

The system has a mass gap equal to $(1 - |\Gamma|)$. In the limit $\Gamma \to \pm 1$, the mass gap disappears and the system falls to a critical point, but in the vicinity of the critical point one can implement the scaling limit $a \to 0$ (a is the lattice parameter) with the corresponding rescaling of the remaining parameters.

Without loss of generality, we can restrict ourselves by the limit of $\Gamma \to 1$. Let $pa = \pi/2 - |k|$. Then for $|p|a \ll 1$ we have

$$\epsilon(p) = 2a\sqrt{m^2 + p^2}, \qquad m = \frac{1-\Gamma}{2a}. \tag{16}$$

The Hamiltonian $H_{FF} = \frac{1}{2a} H_{XY}$ is the Hamiltonian of free Dirac fermions.

Consider the Bogoliubov transformation near the points $\kappa = \pm\pi/2$. Let, for example, $k = \pi/2 - pa$. From (13) we see that for $\Gamma \to 1$ the parameter κ is given by the formula

$$\operatorname{ctg}\kappa = \frac{|p|}{m},$$

and the Bogoliubov transform is trivial everywhere except in the domain $p \sim m$. Of course, we are interested in the region $pa \ll 1$, but in this case the momentum p can be much larger than the mass m. In this area, $\kappa = 0$ and the transformation takes a simple form

$$\begin{aligned} a_{\pi/2-pa} &= ib'^+_{-\pi/2+pa}, \qquad & a_{\pi/2+pa} &= b_{-\pi/2+pa}, \\ a_{-\pi/2+pa} &= ib'^+_{\pi/2-pa}, \qquad & a_{-\pi/2-pa} &= b_{\pi/2-pa}, \qquad m \ll p \ll a^{-1}. \end{aligned} \tag{17}$$

Now we examine the contribution of the interaction $\Delta \neq 0$ in the scaling limit. It is convenient to start from the case $m = 0$ ($\Gamma = 1$). First of all, by means of $b_{\pm\pi/2+pa}$, $b'_{\pm\pi/2+pa}$ we need to introduce the standard relativistic fermion operators. In the notation of lecture 7, the Weyl fermions should have the form

$$\begin{aligned}
\frac{\psi_R(x)}{\sqrt{Na}} &= \int_0^\infty \frac{m_0^{1/2}dp}{2\pi\sqrt{2p}}\left(b_{-\pi/2+pa}\chi_\lambda(x) + b'^+_{-\pi/2+pa}\chi_{\lambda+i\pi}(x)\right)\Big|_{\substack{m_0\to 0\\ m_0e^\lambda=2p}} \\
&= \begin{pmatrix}1\\0\end{pmatrix}\int_0^\infty \frac{dp}{2\pi}\left(b_{-\pi/2+pa}e^{ipx} + ib'^+_{-\pi/2+pa}e^{-ipx}\right) = \begin{pmatrix}1\\0\end{pmatrix}\int_0^\infty \frac{dp}{2\pi}\left(a_{\pi/2+pa}e^{ipx} + a_{\pi/2-pa}e^{-ipx}\right), \\
\frac{\psi_L(x)}{\sqrt{Na}} &= \int_{-\infty}^0 \frac{m_0^{1/2}dp}{2\pi\sqrt{2|p|}}\left(b_{\pi/2+pa}\chi_\lambda(x) + b'^+_{\pi/2+pa}\chi_{\lambda+i\pi}(x)\right)\Big|_{\substack{m_0\to 0\\ m_0e^{-\lambda}=2|p|}} \\
&= \begin{pmatrix}0\\i\end{pmatrix}\int_{-\infty}^0 \frac{dp}{2\pi}\left(b_{\pi/2+pa}e^{ipx} + ib'^+_{\pi/2+pa}e^{-ipx}\right) = \begin{pmatrix}0\\i\end{pmatrix}\int_{-\infty}^0 \frac{dp}{2\pi}\left(a_{-\pi/2+pa}e^{ipx} + a_{-\pi/2-pa}e^{-ipx}\right),
\end{aligned}$$

The complete Dirac fermion has the form

$$\psi(x) = \psi_R(x) + \psi_L(x) = \begin{pmatrix}\psi_+(x)\\ i\psi_-(x)\end{pmatrix},$$

where

$$\psi_\pm(x) = \int_{-\infty}^\infty \frac{dp}{2\pi}\,a_\pm(p)e^{ipx} \qquad a_\pm(p) = (Na)^{1/2}a_{\pm\pi/2+pa}, \tag{18}$$

are written as ordinary nonrelativistic fermions in terms of $a_\pm(p)$ oscillators. Notice, that the initial fermions a_n are expressed in terms of them as

$$a_n = a^{1/2}(i^n\psi_-(an) + i^{-n}\psi_+(an)).$$

Now let us rewrite the Hamiltonian in the form

$$H_\Delta \equiv \frac{1}{2a}(H_{XXZ} + \Delta N/2 - 2\Delta M) = H_{FF} - \frac{\Delta}{Na}\sum_q \rho_q\rho_{-q}\cos q, \tag{19}$$

where

$$M = \rho_0 = \sum_n a_n^+ a_n = \sum_k a_k^+ a_k \tag{20}$$

is the fermion number, and

$$\rho_q = \sum_k a_{k+q}^+ a_k = \sum_n a_n^+ a_n e^{iqn} \tag{21}$$

is the Fourier transform of the corresponding current. Strictly speaking, the fermion number is exactly conserved only at the point $\Gamma = 1$. But it also conserves in the scaling limit, so that subtracting it from the Hamiltonian in this limit does not change anything. The main contribution to the interaction will be made by the region of lowest energies near the points $k = \pm\pi/2$. Therefore, they will have the value ρ_q with the parameter q in the neighborhood of the points 0 and π. Rewrite ρ_q as

$$\rho_{pa} = \rho_{++}(p) + \rho_{--}(p), \qquad \rho_{\pi+pa} = \rho_{+-}(p) + \rho_{-+}(p), \qquad \rho_{\alpha\beta}(p) = \int \frac{dp'}{2\pi}\,a_\alpha^+(p'+p)a_\beta(p'). \tag{22}$$

Substituting it into (19), we obtain

$$\begin{aligned}
H_\Delta = H_{FF} - \Delta\int \frac{dp}{2\pi}(\rho_{++}(p) + \rho_{--}(p))(\rho_{++}(-p) + \rho_{--}(-p)) \\
+ \Delta\int \frac{dp}{2\pi}(\rho_{+-}(p) + \rho_{-+}(p))(\rho_{+-}(-p) + \rho_{-+}(-p)).
\end{aligned}$$

Passing to the coordinate representation, we obtain

$$\begin{aligned} H_\Delta &= H_{FF} - \Delta \int dx\,((\psi_+^+\psi_+)^2 + (\psi_-^+\psi_-)^2 + 2\psi_+^+\psi_+\psi_-^+\psi_- - \psi_-^+\psi_+\psi_+^+\psi_- - \psi_+^+\psi_-\psi_-^+\psi_+) \\ &= H_{FF} - 4\Delta \int dx\, \psi_+^+\psi_+\psi_-^+\psi_-. \end{aligned} \tag{23}$$

Here we used the commutation relations $\psi_\alpha^+(x)\psi_\beta(x) + \psi_\beta(x)\psi_\alpha^+(x) = a^{-1}\delta_{\alpha\beta}$. In particular, $(\psi_\alpha^+\psi_{-\alpha})^2 = (\psi_\alpha^+)^2\psi_{-\alpha}^2 = 0$.

Comparing (23) with the formulas for the Thirring model, we conclude that

$$g = -2\Delta \qquad (\Delta \ll 1). \tag{24}$$

It is clear that this formula is true only for small Δ. It is enough to recall that the mass term $\bar\psi\psi$ in the Thirring model does not have a mass dimension, so the scaling law (16) certainly does not hold for $\Delta \neq 0$.

In deriving these formulas, I deceived you a little. The fact is that the product $\psi_\alpha^+\psi_\alpha$ contains a regular contribution. Indeed, if we bosonize the fermions by using the construction from lecture 2 for $g \to 0$, we obtain

$$\begin{aligned} \psi_+^+(z')\psi_+(z) &= \frac{i}{2\pi}\left(\frac{1}{z'-z} - i\partial\phi(x) + O(z'-z)\right), \\ \psi_-^+(\bar z')\psi_-(\bar z) &= -\frac{i}{2\pi}\left(\frac{1}{\bar z'-\bar z} - i\bar\partial\phi(x) + O(\bar z'-\bar z)\right). \end{aligned}$$

As a result, the terms $(\psi_\alpha^+\psi_\alpha)^2$ contains contributions to the Hamiltonian proportional to $(\partial_x\phi)^2$, which renormalizes the spatial component of the momentum [1].

Is it possible to establish an exact relation between the parameter g of the Thirring model and the parameter Δ of the XYZ model? This is possible since the XYZ model admits an exact solution [2]. Unfortunately, this solution is very complicated, and I cannot give it here. From the solution we can extract the correlation length r_c, which for $\Gamma \to 1$ is proportional to

$$r_c \sim \left(\frac{1-\Delta^2}{1-\Gamma}\right)^{1/(2-2\mu/\pi)}, \qquad \Delta = -\cos\mu. \tag{25}$$

Recalling that $1-\Gamma$ is proportional to the bare mass m_0 in the Thirring model, and that $m_0 \sim m^{2-\beta^2}$, where m is the mass of physical excitations, we obtain

$$\beta^2 = \frac{2\mu}{\pi}, \qquad \frac{g}{\pi} = \frac{\pi/2-\mu}{\mu}. \tag{26}$$

For $\Delta \to 0$ ($\mu \to \pi/2$) the expression (24) is reproduced.

Now let us try to confront operators in the field theoy to operators on the lattice in. For this purpose, it is more convenient to use the sine-Gordon model. Recall that in the Gaussian model (corresponding to the XXZ case) there is a symmetry

$$\phi \to \phi + \alpha,$$

where α is an arbitrary constant. In the sine-Gordon model, this symmetry is broken and α must be an integer multiple of $2\pi/\beta$:

$$\alpha = \frac{2\pi n}{\beta}. \tag{27}$$

The XXZ model also has continuous symmetry

$$\sigma^\pm \to e^{\pm i\lambda}\sigma^\pm, \qquad \sigma^z \to \sigma^z.$$

This symmetry corresponds to the conservation of the z-projection of the full spin

$$S^z = \frac{1}{2}\sum_{n=1}^{N}\sigma_n^z = M - \frac{N}{2}. \tag{28}$$

For $\Gamma < 1$, the term is proportional to

$$\sum_n (\sigma_n^+ \sigma_{n+1}^+ + \sigma_n^- \sigma_{n+1}^-)$$

and corresponding to the mass term in the Thirring model, that is, to the cosine in the sine-Gordon model, violates this symmetry to

$$\lambda = \pi n. \tag{29}$$

By comparing (27) with (29), we conclude that

$$\alpha = \frac{2\lambda}{\beta} \tag{30}$$

Therefore

$$\sigma_n^\pm \sigma_{n+1}^\pm \sim a^{\beta^2} e^{\pm i\beta\phi}.$$

Since the operator $\sigma_n^\pm$ commutes with the operator $\sigma_m^\pm \sigma_{m+1}^\pm$ ($n \neq m, m+1$), the corresponding operator in the field theory must be mutually local to it. It is natural to conjecture that

$$\sigma_n^\pm \sim a^{\beta^2/4} e^{\pm i \frac{\beta}{2}\phi}.$$

General admissible exponential operators in theory have the form

$$V_{m,n}(x) = \exp\left(im\frac{\beta}{2}\phi + in\frac{1}{2\beta}\tilde{\phi}\right), \qquad m, n \in \mathbb{Z}. \tag{31}$$

In particular, the operator σ_n^z is proportional to a linear combination of $a\partial_t\phi$ and $(-1)^n a^{1/\beta^2}(V_{0,2} - V_{0,-2})$. The relationship between lattice and field theory variables is described in great detail in [3].

For the solution of the XXZ model, see the next lecture.

Bibliography

[1] A. Luther and I. Peschel, Phys. Rev. B12 (1975) 3908; A. Luther, Phys. Rev. B14 (1976) 2153.

[2] R. J. Baxger, *Exactly solved models in statistical mechanics*, Academic Press, 1982

[3] S. Lukyanov and V. Terras, Nucl. Phys. B654 (2003) 323 [arXiv:hep-th/0206093].

Problems

1. Derive (8).

2. The XY chain in a magnetic field. Find the spectrum of the Hamiltonian

$$H = -\frac{1}{2}\sum_{n=1}^N (\sigma_n^x \sigma_{n+1}^x + \Gamma \sigma_n^y \sigma_{n+1}^y + 2h\sigma_n^z).$$

Find the critical values of the external field h depending on Γ, at which the gap in the spectrum disappears.

3. Analogously to the reasoning (16)"– (21) investigate the case $\Gamma \to -1$.

4. Show that all the operators (31) are mutually local or mutually semilocal, that is, any product $V_{m_1 n_1}(x_1) V_{m_2 n_2}(x_2)$ does not change when passing x_2 around x_1 in the complex plane, or multiplies by -1.

5*. Under the conditions of Problem 2 describe the system in the vicinity of critical lines. Show that the critical behavior for $|\Gamma| < 1$ is described by a free *Majorana* fermion.