Lecture 7
Thirring model: solution by the Bethe Ansatz method

Consider the massive Thirring model

$$S^{MT}[\psi,\bar\psi]=\int d^2x\,\left(\bar\psi(i\hat\partial-m_0)\psi-\frac{g}{2}(\bar\psi\gamma^\mu\psi)^2\right) \tag{1}$$

with

$$\psi=\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix},\qquad \gamma^0=\begin{pmatrix} & -i\\ i & \end{pmatrix}=\sigma^2,\qquad \gamma^1=\begin{pmatrix} & i\\ i & \end{pmatrix}=i\sigma^1,\qquad \gamma^3=\gamma^0\gamma^1=\begin{pmatrix}1 & \\ & -1\end{pmatrix}=\sigma^3. \tag{2}$$

The Hamiltonian of the Thirring model has the form

$$H=\int dx\,\left(-i\psi^+\sigma^3\partial_x\psi+m_0\psi^+\sigma^2\psi+2g\psi_+^+\psi_-^+\psi_-\psi_+\right) \tag{3}$$

with the commutation relations

$$\psi^+_{\alpha'}(x')\psi_\alpha(x)+\psi_\alpha(x)\psi^+_{\alpha'}(x')=\delta_{\alpha'\alpha}\delta(x'-x), \tag{4}$$

while the momentum P and the fermion number operator Q have the form

$$P=-i\int dx\,\psi^+\partial_x\psi,\qquad Q=\int dx\,\psi^+\psi. \tag{5}$$

Let us recall Dirac's picture. The spectrum $\epsilon^2-p^2=m^2$ has two branches: $\epsilon=\pm\sqrt{p^2+m^2}$. According to the Pauli principle only one excitation can reside one state. The state of the system, in which all one-particle states are empty, we will call a "bare vacuum" or a *pseudovacuum*. If we begin to fill the states with negative energy with fermions, the energy of the system decreases. Thus the pseudovacuum is not the ground state of the system. The ground state (the physical vacuum) will appear when we fill all states of negative energy ("Dirac's sea"). "Elementary excitations" above the pseudovacuum will be referred to as *pseudoparticles*. Let us try to formalize this procedure.

We first consider the case of free fermions $g=0$. Denote by $|\Omega\rangle$ the state that satisfies the conditions

$$\psi_\alpha(x)|\Omega\rangle=0,\qquad \langle\Omega|\psi^+_\alpha(x)=0. \tag{6}$$

Introduce the wave function of a state of N pseudoparticles:

$$|\chi_N\rangle=\int d^Nx\,\chi^{\alpha_1\ldots\alpha_N}(x_1,\ldots,x_N)\psi^+_{\alpha_N}(x_N)\ldots\psi^+_{\alpha_1}(x_1)|\Omega\rangle. \tag{7}$$

States of this form are eigenvectors of the operator Q:

$$Q|\chi_N\rangle=N|\chi_N\rangle.$$

Thus, the total space of states $\mathcal H$ splits into a sum over eigenvalues of Q:

$$\mathcal H\simeq\bigoplus_{N=0}^{\infty}\mathcal H_N,\qquad v\in\mathcal H_N\Leftrightarrow Qv=Nv. \tag{8}$$

The fermion number operator becomes the operator of the number of pseudoparticles in this picture.

The action $\hat H_N$ of the Hamiltonian on the wave function $\chi^{\alpha_1\cdots\alpha_N}(x_1,\ldots,x_N)$, defined by the equation

$$H|\chi_N\rangle=\int d^Nx\,(\hat H_N\chi)^{\alpha_1\ldots\alpha_N}(x_1,\ldots,x_N)\psi^+_{\alpha_N}(x_N)\ldots\psi^+_{\alpha_1}(x_1)|\Omega\rangle,$$

has the form

$$\hat H_N=\sum_{k=1}^{N}(-i\sigma^3_k\partial_{x_k}+m_0\sigma^2_k),$$

where σ_k^i acts on the space of the kth particle. For $N = 1$, the eigenstate has the form

$$\chi_\lambda(x) = \begin{pmatrix} e^{\lambda/2} \\ ie^{-\lambda/2} \end{pmatrix} e^{ixm_0 \operatorname{sh} \lambda}. \tag{9}$$

The many-particle solution of the free-field Hamiltonian is given by the Slater determinant:

$$\chi^{\alpha_1 \dots \alpha_N}_{\lambda_1 \dots \lambda_N}(x_1, \dots, x_N) = \sum_\sigma (-1)^\sigma \prod_{k=1}^N \chi^{\alpha_{\sigma_k}}_{\lambda_k}(x_{\sigma_k}). \tag{10}$$

The energy of the N-particle state is equal to

$$E_N(\lambda_1, \dots, \lambda_N) = m_0 \sum_{k=1}^N \operatorname{ch} \lambda_k. \tag{11}$$

What values can the λ parameters take? If the system is in a box of size L with the cyclic boundary conditions, the "rapidities" λ_k are solutions to the equations

$$e^{im_0 L \operatorname{sh} \lambda_k} = 1, \qquad k = 1, \dots, N. \tag{12}$$

Therefore,

$$\operatorname{sh} \lambda_k = \frac{2\pi n_k}{m_0 L}, \qquad n_k \in \mathbb{Z}.$$

This means that λ_k lies either on the real axis $\mathbb{R}$, or on the line $i\pi + \mathbb{R}$. The latter solutions correspond to negative energies. Obviously, the ground state is the state in which all states of negative energy are filled. Let $\lambda_k = i\pi + \xi_k$. To determine the vacuum energy, we introduce an ultraviolet cutoff

$$-\Theta < \xi_k < \Theta, \qquad \Theta \simeq \log \frac{\Lambda}{m_0}. \tag{13}$$

In the thermodynamic limit $L \to \infty$, the vacuum energy is equal to

$$E_0 = -L \int_{-\Theta}^{\Theta} \frac{d\xi}{2\pi} \rho(\xi) m_0 \operatorname{ch} \xi, \qquad \rho(\xi) = \frac{2\pi}{L} \left| \frac{dn}{d\xi} \right| = m_0 \operatorname{ch} \xi.$$

Of course, the energy of the ground state is meaningless by itself, the energies of excited states are of interest. The excitation with the rapidity θ corresponds to an additional root at the point

$$\lambda_k = \theta \qquad \text{(particle)},$$

or to a hole (lack of a root) at the point

$$\lambda_k = \theta + i\pi \qquad \text{(antiparticle)}.$$

Since the roots of the equations (12) are unrelated to each other, we get a system of non-interacting particles and antiparticles with $p = (m \operatorname{ch} \theta, m \operatorname{sh} \theta)$ that obey the Pauli principle, that is, what we should have obtained.

Now turn on the interaction. The interaction operator in (3) commutes with the pseudoparticle number operator Q. Thus, the interaction operator acts inside the spaces $\mathcal{H}_N$:

$$\hat{H}_N = \sum_{k=1}^N (-i\sigma_k^3 \partial_{x_k} + m_0 \sigma_k^2) + g \sum_{k<l}^N \delta(x_k - x_l)(1 - \sigma_k^3 \sigma_l^3). \tag{14}$$

The construction of sigma-matrices on the right side is:

$$\frac{1}{2}(1 \otimes 1 - \sigma^3 \otimes \sigma^3)^{\alpha_1' \alpha_2'}_{\alpha_1 \alpha_2} = \delta^{\alpha_1'}_{\alpha_1} \delta^{\alpha_2'}_{\alpha_2} \delta_{\alpha_1, -\alpha_2}, \tag{15}$$

which means that only the particles of opposite spins interact.

The interaction term in the Hamiltonian (14) is poorly defined. Indeed, the eigenfunction equation is a first order differential equation. Thus the delta-functional term results in a discontinuity of the wave function is on the surface $x_k = x_l$. At the same time, the action of the Hamiltonian depends on the wave function just at this surface. The delta-function should be regularized. We show that the answer does not depend on regularization. Consider the equation

$$f'(x) - c\delta(x)f(x) = g(x, f(x))$$

Let us regularize the delta-function by an arbitrary integrable function $\delta_a(x)$ with the support $[-a, a]$:

$$f'(x) - c\delta_a(x)f(x) = g(x, f(x)), \qquad \int_{-\infty}^{\infty} dx\, \delta_a(x) = 1. \tag{16}$$

Let

$$\delta_a(x) = \epsilon'_a(x).$$

Then

$$\frac{f'(x)}{f(x)} = c\epsilon'_a(x) + \frac{g(x, f(x))}{f(x)}.$$

For small enough a the second term may be neglected, and we have

$$f(x) = \text{const}\, e^{c\epsilon_a(x)} \qquad \Rightarrow \qquad f(+a) = e^{c(\epsilon_a(a)-\epsilon_a(-a))}f(-a) = e^c f(-a).$$

Hence, in the limit $a \to 0$ we obtain

$$f(+0) = e^c f(-0). \tag{17}$$

Single-particle states are again described by solutions (9). Consider a two-particle state. Since the interaction is contact (nonzero only for $x_1 = x_2$), for $x_1 \neq x_2$ the wave function is a solution to the equations for free fermions. Thanks to the conservation laws of energy and momentum the scattering is reflectionless, and the wave function reads

$$\chi^{\alpha_1\alpha_2}_{\lambda_1\lambda_2}(x_1, x_2) = \begin{cases} A_{12}\chi^{\alpha_1}_{\lambda_1}(x_1)\chi^{\alpha_2}_{\lambda_2}(x_2) - A_{21}\chi^{\alpha_1}_{\lambda_2}(x_1)\chi^{\alpha_2}_{\lambda_1}(x_2) & \text{for } x_1 < x_2, \\ A_{21}\chi^{\alpha_1}_{\lambda_1}(x_1)\chi^{\alpha_2}_{\lambda_2}(x_2) - A_{12}\chi^{\alpha_1}_{\lambda_2}(x_1)\chi^{\alpha_2}_{\lambda_1}(x_2) & \text{for } x_1 > x_2. \end{cases} \tag{18}$$

This function is evidently antisymmetric in (α_1, x_1), (α_2, x_2) and contains (up to a normalization) one free parameter A_{12}/A_{21}, which should depend on the coupling constant g. A direct calculation with the aid of (17) gives

$$\frac{A_{21}}{A_{12}} = R(\lambda_1 - \lambda_2), \qquad R(\lambda) = e^{i\Phi(\lambda)} = \frac{\operatorname{ch}\frac{\lambda - ig}{2}}{\operatorname{ch}\frac{\lambda + ig}{2}}. \tag{19}$$

The function $R(\lambda)$ has the meaning of the scattering matrix of pseudoparticles. It is convenient to fix the scattering phase $\Phi(\lambda)$ by the skew symmetry condition

$$\Phi(-\lambda) = -\Phi(\lambda), \tag{20}$$

assuming that the cuts lie on the rays $(i(\pi - |g|), i\infty)$, $(-i(\pi - |g|), -i\infty)$. The quantities λ_k are naturally defined modulo $2\pi i$.

Note that the function $R(\lambda)$ is periodic in g with the period 2π. Since we will build the vacuum close to the vacuum of free fermions, it should be assumed that the solution makes sense for

$$-\pi < g < \pi. \tag{21}$$

Now it is easy to construct a general N-particle solution (*Bethe Ansatz*):

$$\chi^{\alpha_1\ldots\alpha_N}_{\lambda_1\ldots\lambda_N}(x_1, \ldots, x_N) = \sum_\tau (-1)^{\sigma\tau} A_\tau \prod_{k=1}^{N} \chi^{\alpha_{\sigma_k}}_{\lambda_{\tau_k}}(x_{\sigma_k}) \quad \text{for } x_{\sigma_1} < \ldots < x_{\sigma_N}. \tag{22}$$

The coefficients A satisfy the relations

$$A_{\dots,i+1,i,\dots} = R(\lambda_i - \lambda_{i+1}) A_{\dots,i,i+1,\dots}. \tag{23}$$

Impose a cyclic boundary condition

$$\chi(\dots, x_k + L, \dots) = \chi(\dots, x_k, \dots). \tag{24}$$

We obtain

$$e^{im_0 L \operatorname{sh} \lambda_k} \prod_{\substack{l=1 \\ l \neq k}}^{N} R(\lambda_k - \lambda_l) = 1. \tag{25}$$

This system of equations for the parameters λ_k is called the system of *Bethe equations*. Bethe equations are nonlinear equations with N unknowns, moreover, as expected, in physically interesting cases $N \to \infty$. Nevertheless, a huge step has been taken: solving the problem is reduced to solving a system of algebraic equations. Each solution to this system, i.e. each set (up to a permutation) of numbers $(\lambda_1, \dots, \lambda_N)$, satisfying (25), corresponds to a single state of the system. The components λ_k of the solution are called *roots* of the Bethe equations.

By taking the logarithm of the Bethe equations, we find

$$m_0 L \operatorname{sh} \lambda_k + \sum_{l=1}^{N} \Phi(\lambda_k - \lambda_l) = 2\pi n_k, \tag{26}$$

and the energy and momentum of the state are equal to

$$E_N(\lambda_1, \dots, \lambda_N) = m_0 \sum_{i=1}^{N} \operatorname{ch} \lambda_i, \qquad P_N(\lambda_1, \dots, \lambda_N) = m_0 \sum_{i=1}^{N} \operatorname{sh} \lambda_i. \tag{27}$$

We ask ourselves: how can roots of the Bethe equations be located? Naturally, real roots and roots on the line $i\pi + \mathbb{R}$ are suitable. General complex roots can be located in pairs symmetrically with respect to one of the lines $\mathbb{R}$ and $i\pi + \mathbb{R}$. A more detailed analysis shows that there can be no other types of solutions.

The Bethe equations map each solution $(\lambda_1, \dots, \lambda_N)$ on a set of numbers $(n_1, \dots, n_N)$. It can be shown that $n_k = n_l$ only if $\lambda_k = \lambda_l$. It follows from the Pauli principle that all λ_k must be different and that, therefore,

$$n_k \neq n_l \qquad (k \neq l). \tag{28}$$

It is natural to conjecture that the solution of the Bethe equations with negative energies of "bare particles", i.e. with $\operatorname{Im} \lambda_k = \pi$, corresponds to the least energy. We will write

$$\lambda_k = i\pi + \xi_k.$$

To minimize the energy, it is necessary to fill all the states of negative energy, therefore it is necessary that the integers n_k were consecutive integers:

$$n_{k+1} - n_k = \pm 1, \tag{29}$$

and it is convenient to choose the sign so that the value ξ_k grew with k. Therefore we assume

$$n_k = k_0 - k$$

with a certain k_0. Then

$$m_0 L \operatorname{sh} \xi_k = 2\pi(k - k_0) + \sum_{l=1}^{N} \Phi(\xi_k - \xi_l). \tag{30}$$

In the thermodynamic limit $L \to \infty$ the distance between the levels tends to zero, and this equation can be differentiated with respect to ξ_k. We will obtain

$$m_0 \operatorname{ch} \xi = \rho(\xi) + \int_{-\Theta}^{\Theta} \frac{d\xi'}{2\pi} \Phi'(\xi - \xi') \rho(\xi'). \tag{31}$$

Here

$$\rho(\xi) = \frac{2\pi}{L}\frac{dk}{d\xi} \tag{32}$$

is the spectral density of states related with the number of pseudoparticles N in a state by the formula

$$\int_{-\Theta}^{\Theta} \frac{d\xi}{2\pi}\,\rho(\xi) = \frac{N}{L}. \tag{33}$$

For $\Theta \to \infty$ the value of m_0 tends to zero for $g > 0$ and to infinity for $g < 0$. For $g > 0$ it can be shown that $\rho(\xi)/m_0 \to \infty$. Therefore, you should look for a formal solution to the homogeneous equation, which is obtained from (31) for $m_0 = 0$, $\Theta \to \infty$. It has the form

$$\rho(\xi) = \mathrm{const}\cdot \mathrm{ch}\,\frac{\pi\xi}{\pi+g}. \tag{34}$$

The proportionality coefficient can be found by a more accurate calculation, and it turns out to be finite.

Consider now the fermion sea with holes.[1] To do this, let us generalize the equation (30):

$$m_0 L\,\mathrm{sh}\,\xi_k = -2\pi n_k + \sum_{l=1}^{N}\Phi(\xi_k - \xi_l). \tag{35}$$

Next, let us define $\xi(n)$ by the equation

$$m_0 L\,\mathrm{sh}\,\xi(n) = -2\pi n + \sum_{l=1}^{N}\Phi(\xi(n) - \xi_l) \tag{36}$$

Let us determine the density of states $\rho(\xi)$ and the density of holes $\rho^\circ(\xi) = \rho(\xi) - \rho^\bullet(\xi)$ as follows:

$$\rho(\xi(n)) = \frac{2\pi}{L|\xi(n+1)-\xi(n)|} \simeq \frac{2\pi}{L}\left|\frac{dn}{d\xi(n)}\right|, \qquad \rho^\bullet(\xi) = \left\langle \frac{2\pi}{L|\xi_{k+1}-\xi_k|}\right\rangle_{\xi_k\simeq\xi} = \left\langle \frac{2\pi}{L}\left|\frac{dk}{d\xi_k}\right|\right\rangle_{\xi_k\simeq\xi}. \tag{37}$$

In particular, for one hole with the parameter $\xi = \xi_0$ we have $\rho^\circ(\xi) = 2\pi L^{-1}\delta(\xi-\xi_0)$. Then the system of equations looks as follows:

$$m_0\,\mathrm{ch}\,\xi = \rho(\xi) + \int_{-\Theta}^{\Theta}\frac{d\xi'}{2\pi}\,\Phi'(\xi-\xi')(\rho(\xi') - \rho^\circ(\xi')). \tag{38}$$

Denoting by $\rho_0(\xi)$ the solution to the equation (31) and subtracting this equation from (38), we obtain

$$\delta\rho(\xi) + \int_{-\Theta}^{\Theta}\frac{d\xi'}{2\pi}\,\Phi'(\xi-\xi')\delta\rho(\xi') = \int_{-\Theta}^{\Theta}\frac{d\xi'}{2\pi}\,\Phi'(\xi-\xi')\rho^\circ(\xi'). \tag{39}$$

Here

$$\delta\rho(\xi) = \rho(\xi) - \rho_0(\xi).$$

In the limit $\Theta \to \infty$ this equation is easily solved by the Fourier method. Indeed, let

$$\tilde\Phi'(\omega) = \int_{-\infty}^{\infty}\frac{d\xi}{2\pi}\,\Phi'(\xi)e^{i\xi\omega}, \qquad \delta\tilde\rho(\omega) = \int_{-\infty}^{\infty}\frac{d\xi}{2\pi}\,\delta\rho(\xi)e^{i\xi\omega}, \qquad \tilde\rho^\circ(\omega) = \int_{-\infty}^{\infty}\frac{d\xi}{2\pi}\,\rho^\circ(\xi)e^{i\xi\omega}.$$

Applying the Fourier transform to the equation (39), we obtain the algebraic equation

$$\delta\tilde\rho(\omega) + \tilde\Phi'(\omega)\delta\tilde\rho(\omega) = \Phi'(\omega)\tilde\rho^\circ(\omega).$$

[1] Unfortunately, an error was made in the initial version of the lecture when considering the sea of fermions with one additional particle. I am grateful to I. Protopopov, who noticed it while solving a problem. A study of an additional particle, in fact, requires consideration of the so-called string solutions of the Bethe equations.

It is easy to check that

$$\tilde{\Phi}'(\omega) = -\frac{\operatorname{sh} g\omega}{\operatorname{sh} \pi\omega}, \qquad \delta\tilde{\rho}(\omega) = -\frac{\operatorname{sh} g\omega}{2 \operatorname{sh} \frac{\pi-g}{2}\omega \operatorname{ch} \frac{\pi+g}{2}\omega}\tilde{\rho}^\circ(\omega). \tag{40}$$

We begin the study of the solution (40) from calculating the charge of the excitation that corresponds to a hole. It would seem that the hole charge should be equal to $N - N_0 = -1$. To make sure this is not the case, take into account the ultraviolet cutoff. When the density of holes changes, the density of states also changes, so that the number of states under the cutoff, that is, in the interval $-\Theta < \xi < \Theta$ also changes. We will be interested in two quantities:

$$\begin{aligned} \Delta N &= -L\int_{-\Theta}^{\Theta} \frac{d\xi}{2\pi}\,\rho^\circ(\xi) = -L\tilde{\rho}^\circ(0), \\ \Delta Q &= L\int_{-\Theta}^{\Theta} \frac{d\xi}{2\pi}\,(\delta\rho(\xi) - \rho^\circ(\xi)). \end{aligned} \tag{41}$$

In the first line we used the assumption that all holes are under the cutoff, that is, $\rho^\circ(\xi) = 0$ for $|\xi| > \Theta$. The ratio

$$z^\circ = -\frac{\Delta Q}{\Delta N} \tag{42}$$

gives the charge of a hole. By computing

$$\Delta Q = -L\int_{-\Theta}^{\Theta} \frac{d\xi}{2\pi}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega\xi}\frac{\operatorname{sh} \pi\omega}{2 \operatorname{sh} \frac{\pi-g}{2}\omega \operatorname{ch} \frac{\pi+g}{2}\omega}\tilde{\rho}^\circ(\omega) \simeq \frac{\pi}{\pi - g}\Delta N, \tag{43}$$

we find

$$z^\circ = -\frac{\pi}{\pi - g}. \tag{44}$$

We see that the charge of a hole is not an integer in terms of pseudoparticles. In other words, the particle charge is renormalized.

This renormalization has a significant consequence. To understand it, let us return to the derivation of the Hamiltonian (3). We formally derived it according to classical rules from the classical action. The quantum effect is that one physical particle amounts $|z^\circ|$ pseudoparticles defined in (7):

$$Q = |z^\circ|\int dx\, \psi^+_{\text{phys}}\psi_{\text{phys}} + \text{const}\,, \tag{45}$$

where ψ_{phys} are physical fields, that is, just those fields that were discussed in lecture 2. It means that

$$\psi = |z^\circ|^{1/2}\psi_{\text{phys}}. \tag{46}$$

Substituting it into the Hamiltonian (3), we see that the physical constant g_{phys} (which was denoted as g in lecture 2) is related to the formal constant g as

$$g_{\text{phys}} = g|z^\circ| = \frac{g}{1 - g/\pi} \quad \Leftrightarrow \quad \frac{1}{g_{\text{phys}}} = \frac{1}{g} - \frac{1}{\pi}. \tag{47}$$

If the formal coupling constant varies within $-\pi \le g < \pi$, then the physical one varies within $-\frac{\pi}{2} \le g_{\text{phys}} < \infty$ in accordance with the results of the bosonization.

The energy $E[\rho^\circ]$ and the momentum $P[\rho^\circ]$ of the system are functionals of the hole density ρ°, and the excitation energy is defined as the difference $E[\rho^\circ] - E[0]$. We have

$$\begin{aligned} E[\rho^\circ] - E[0] &= m_0 L\int_{-\Theta}^{\Theta} \frac{d\xi}{2\pi}\,(\rho^\circ(\xi) - \delta\rho(\xi)) \operatorname{ch}\xi, \\ P[\rho^\circ] &= m_0 L\int_{-\Theta}^{\Theta} \frac{d\xi}{2\pi}\,(\rho^\circ(\xi) - \delta\rho(\xi)) \operatorname{sh}\xi. \end{aligned}$$

The situation here is different for positive and for negative g.

For $g < 0$ the integrals in these expressions converge as $\Theta \to \infty$. But if we put $\Theta = \infty$, we obtain that $E[\rho^\circ] - E[0] = P[\rho^\circ] = 0$, since $\delta\tilde{\rho}(\pm i) = \tilde{\rho}^\circ(\pm i)$. Therefore, the bare mass m_0 should be renormalized. An explicit calculation of the first correction related to the poles of the function $\delta\tilde{\rho}(\omega)$ at the points $\omega = \pm\frac{i\pi}{\pi+g}$ gives

$$E[\rho^\circ] - E[0] = L\int_{-\infty}^{\infty} \frac{d\xi}{2\pi}\,\epsilon(\xi)\rho^\circ(\xi), \qquad P[\rho^\circ] = L\int_{-\infty}^{\infty} \frac{d\xi}{2\pi}\,p(\xi)\rho^\circ(\xi) \tag{48}$$

where

$$\epsilon(\lambda) = m\,\mathrm{ch}\,\frac{\pi\lambda}{\pi+g}, \qquad p(\lambda) = m\,\mathrm{sh}\,\frac{\pi\lambda}{\pi+g}, \qquad m = \frac{M}{g}\,\mathrm{ctg}\left(\frac{\pi}{2}\frac{\pi-g}{\pi+g}\right), \tag{49}$$

and the constant M is defined by the equality

$$m_0 = M\exp\left(-\frac{g}{\pi+g}\Theta\right) \sim M\left(\frac{m_0}{\Lambda}\right)^{g/(\pi+g)}. \tag{50}$$

Thus, the particles have a relativistic spectrum with the rapidity

$$\theta = \frac{\pi\xi}{\pi+g}. \tag{51}$$

Comparing (50) with our previous estimation

$$m_0 \sim M^{2-\beta^2}, \tag{52}$$

where β is the coupling constant of the sine-Gordon model, we obtain the relation

$$\frac{g}{\pi} = 1 - \beta^2.$$

Recalculating the physical coupling constant, we obtain the relation

$$\frac{g_{\mathrm{phys}}}{\pi} = \beta^{-2} - 1,$$

given in Lecture 2. The formal coupling constant g and the physical coupling constant g_{phys} coincide in the first order in perturbation theory, but differ in the higher orders. This difference must be taken into account when interpreting accurate results.

For $g > 0$ the situation is somewhat different. The poles of the function $\delta\tilde{\rho}(\omega)$ for $\omega = \pm i\pi/(\pi+g)$ become closer to the real axis than the points $\pm i$. This means that the integrals in the formulas for the momentum and the energy diverge, which corresponds to $m_0 \to 0$. Strictly speaking, an explicit calculation of $\delta\rho(\xi)$ and then of the integrals for the energy and the momentum of excitations is required. However, this whole procedure leads to answers obtained by analytic continuation from the region $g < 0$. This means that the formula for the mass renormalization (50) is valid in this case as well.

What else can be obtained from the formulas (40)? It turns out that the hole scattering matrix is immediately extracted from them. To do this, consider the "spinless" fermions of mass m with the scattering matrix $S(\theta) = e^{i\Psi(\theta)}$, $\Psi(-\theta) = -\Psi(\theta)$. Suppose that these fermions live in a space of length L with cyclic boundary conditions. In exactly the same way as (25) we obtain

$$e^{imL\,\mathrm{sh}\,\theta_k} \prod_{\substack{l=1\\ l\neq k}}^{N} S(\theta_l - \theta_k) = 1. \tag{53}$$

Take logarithm of the equations:

$$mL\,\mathrm{sh}\,\theta_k + \sum_{l=1}^{N} \Psi(\theta_k - \theta_l) = 2\pi n_k. \tag{54}$$

and make the thermodynamic limit. To do this, let us define $\theta(n)$ by the equation

$$mL\operatorname{sh}\theta(n)=\sum_{l=1}^{N}\Psi(\theta(n)-\theta_l)=2\pi n. \tag{55}$$

Then arrange n_k in the ascending order and set

$$\rho_*(\theta(n))=\frac{2\pi}{L|\theta(n+1)-\theta(n)|}\simeq\frac{2\pi}{L}\left|\frac{dn}{d\theta(n)}\right|,\qquad \rho^\bullet_*(\theta)=\left\langle\frac{2\pi}{L|\theta_{k+1}-\theta_k|}\right\rangle_{\theta_k\simeq\theta}=\left\langle\frac{2\pi}{L}\left|\frac{dk}{d\theta_k}\right|\right\rangle_{\theta_k\simeq\theta}. \tag{56}$$

The quantity $\rho_*(\theta)$ has the meaning of density of states, while the value $\rho^\bullet_*(\theta)$ has the meaning of particle density. Then the equation for the density of states takes the form

$$m\operatorname{ch}\theta+\int_{-\infty}^{\infty}\frac{d\theta'}{2\pi}\,\Psi'(\theta-\theta')\rho^\bullet_*(\theta')=2\pi\rho_*(\theta). \tag{57}$$

At zero particle density $\rho^\bullet_*$ we have $\rho_{*0}(\theta)=m\operatorname{ch}\theta$. Assuming $\delta\rho_*=\rho_*-\rho_{*0}$, we have

$$\delta\rho_*(\theta)=\int_{-\infty}^{\infty}\frac{d\theta'}{2\pi}\,\Psi'(\theta-\theta')\rho^\bullet_*(\theta'). \tag{58}$$

We pass to the Fourier transforms

$$\delta\tilde\rho_*(\omega)=\int\frac{d\theta}{2\pi}\,\delta\rho_*(\theta)e^{i\theta\omega}\quad\text{etc.}$$

The equation (58) takes the form

$$\delta\tilde\rho_*(\omega)=\tilde\Psi'(\omega)\rho^\bullet_*(\omega). \tag{59}$$

Now suppose that the auxiliary fermions are nothing but our holes. Given (51) we identify

$$\delta\rho_*(\theta)=\alpha\delta\rho(\alpha\theta),\qquad \rho^\bullet_*(\theta)=\alpha\rho^\circ(\alpha\theta),\qquad \alpha=1+\frac{g}{\pi}=2-\beta^2. \tag{60}$$

We have

$$\delta\tilde\rho_*(\omega)=\tilde\rho(\alpha^{-1}\omega),\qquad \delta\tilde\rho^\bullet_*(\omega)=\tilde\rho^\circ(\alpha^{-1}\omega). \tag{61}$$

Under this assumption, comparing (61) with (40), we obtain

$$\Psi(\theta)=i\int_{-\infty}^{\infty}\frac{d\omega}{\omega}\frac{\operatorname{sh}\frac{\pi\omega}{2}\operatorname{sh}\frac{\pi(p-1)\omega}{2}}{\operatorname{sh}\pi\omega\operatorname{sh}\frac{\pi p\omega}{2}}e^{-i\theta\omega}=2\int_0^{\infty}\frac{d\omega}{\omega}\frac{\operatorname{sh}\frac{\pi\omega}{2}\operatorname{sh}\frac{\pi(p-1)\omega}{2}}{\operatorname{sh}\pi\omega\operatorname{sh}\frac{\pi p\omega}{2}}\sin\theta\omega, \tag{62}$$

where the parameter p is defined by the relation

$$\beta^2=2\frac{p}{p+1}.$$

The function $S(\theta)=e^{i\Psi(\theta)}$ is actually the scattering matrix for only one type of particles, antifermions in the massive Thirring model. We need to find the S-matrix in the form of a 4×4 matrix in the basis $(++,+-,-+,--)$ (“$+$” corresponds to the fermions, and “$-$” to the antifermions):

$$S(\theta)=\left(S^{\alpha_1\alpha_2}_{\beta_1\beta_2}(\theta)\right)=\begin{pmatrix}a(\theta)&&&\\&b(\theta)&c(\theta)&\\&c(\theta)&b(\theta)&\\&&&a(\theta)\end{pmatrix}. \tag{63}$$

Here $a(\theta)=e^{i\Psi(\theta)}$ corresponds to the scattering of particles of the same kind and can also be written as

$$a(\theta)=\frac{\Gamma\left(\frac{1}{p}\right)\Gamma\left(1+\frac{i\theta}{\pi p}\right)}{\Gamma\left(\frac{1}{p}+\frac{i\theta}{\pi p}\right)}\prod_{n=1}^{\infty}\frac{R_n(\theta)R_n(i\pi-\theta)}{R_n(0)R_n(i\pi)},\qquad R_n(\theta)=\frac{\Gamma\left(\frac{2n}{p}+\frac{i\theta}{\pi p}\right)\Gamma\left(1+\frac{2n}{p}+\frac{i\theta}{\pi p}\right)}{\Gamma\left(\frac{2n+1}{p}+\frac{i\theta}{\pi p}\right)\Gamma\left(1+\frac{2n-1}{p}+\frac{i\theta}{\pi p}\right)}. \tag{64}$$

The ratios of the coefficients $b(\theta)/a(\theta)$ and $c(\theta)/a(\theta)$ can be found by solving the Young–Baxter equation in conjunction with the crossing symmetry equation. The answer is

$$\frac{b(\theta)}{a(\theta)}=\frac{\operatorname{sh}\frac{\theta}{p}}{\operatorname{sh}\frac{i\pi-\theta}{p}},\qquad \frac{c(\theta)}{a(\theta)}=\frac{\operatorname{sh}\frac{i\pi}{p}}{\operatorname{sh}\frac{i\pi-\theta}{p}}. \tag{65}$$

The formula (64) makes it easy to find the singularities of the functions $a(\theta)$, $b(\theta)$ and $c(\theta)$ on the imaginary axis. A singularity in the interval $(0,i\pi)$ (i.e. on the physical sheet) corresponds to a bound state, if the sign of its residue in $i\theta$ is negative. For $0<p<1$ $(0<\beta^2<1)$ there are such poles in $b(\theta)$ and $c(\theta)$:

$$\theta_n=i\pi-i\pi pn,\qquad n=1,2,\dots,\left\lfloor\frac{1}{p}\right\rfloor. \tag{66}$$

This corresponds to neutral bound states (breathers in the sine-Gordon model) with masses

$$M_n=2m\sin\frac{\pi pn}{2} \tag{67}$$

Bibliography

[1] M. Gaudin, The Bethe wavefunction, Cambridge University Press, 2014 (La fonction d'onde de Bethe, Dunod, Paris, 1983)

[2] N. M. Bogolyubov, A. G. Izergin, V. E. Korepin, Correlation Functions of Integrable Systems and Quantum Method of the Inverse Problem, Nauka, Moscow, 1992.

Problems

1. Obtain (3) from (1). Derive (14).
2. Obtain (9).
3. Obtain (19).
4. Obtain the relations (48)–(49).
5*. By using the Bethe Ansatz method, find the wave function of a system of N identical nonrelativistic bosons interacting due to the potential

$$U(x)=c\delta(x).$$

Find a system of Bethe equations for them Find the spectrum of the Hamiltonian in two limits: $c\to\infty$ and $c\to+0$.