Lecture 7
Thirring model: solution by the Bethe Ansatz method

Thirring model: the Hamiltonian formulation

The action

$$S^{MT}[\psi,\bar\psi] = \int d^2x \, \left(\bar\psi(i\hat\partial - m_0)\psi - \frac{g}{2}(\bar\psi\gamma^\mu\psi)^2\right) \tag{1}$$

with

$$\psi = \begin{pmatrix}\psi_+ \\ \psi_-\end{pmatrix}, \qquad \gamma^0 = \begin{pmatrix} & -i \\ i & \end{pmatrix} = \sigma^2, \qquad \gamma^1 = \begin{pmatrix} & i \\ i & \end{pmatrix} = i\sigma^1. \tag{2}$$

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Conserved charges: the momentum P and the particle number operator Q are

$$P = -i\int dx\,\psi^+\partial_x\psi, \qquad Q = \int dx\,\psi^+\psi. \tag{5}$$

Free fermion. Dirac's picture

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"N-particle" state:

$$|\chi_N\rangle = \int d^N x \, \chi^{\alpha_1 \dots \alpha_N}(x_1, \dots, x_N)\psi^+_{\alpha_1}(x_1)\dots\psi^+_{\alpha_N}(x_N)|\Omega\rangle. \tag{7}$$

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$$Q|\chi_N\rangle = N|\chi_N\rangle \quad \Rightarrow \quad \mathcal{H} \simeq \bigoplus_{N=0}^{\infty} \mathcal{H}_N, \qquad \mathcal{H}_N = \{v \in \mathcal{H} \,|\, Qv = Nv\}\,. \tag{8}$$

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Define action $\hat{H}_N$ of H on the wave function:

$$H|\chi_N\rangle = \int d^N x\, (\hat{H}_N\chi)^{\alpha_1\ldots\alpha_N}(x_1,\ldots,x_N)\psi^+_{\alpha_1}(x_1)\ldots\psi^+_{\alpha_N}(x_N)|\Omega\rangle.$$

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Explicitly, we have

$$\hat{H}_N = \sum_{k=1}^{N}(-i\sigma^3_k\partial_{x_k} + m_0\sigma^2_k),$$

where σ^i_k acts on the space of the kth particle:

$$(\sigma^i_k\chi)^{\alpha_1\ldots\alpha_k\ldots\alpha_N} = \sum_{\alpha'_k}(\sigma^i)^{\alpha_k}_{\alpha'_k}\chi^{\alpha_1\ldots\alpha'_k\ldots\alpha_N}$$

For $N = 1$, the eigenfunction is

$$\chi_\lambda(x) = \begin{pmatrix} e^{\lambda/2} \\ ie^{-\lambda/2} \end{pmatrix} e^{ixm_0 \operatorname{sh} \lambda}. \tag{9}$$

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For general N we have the Slater determinant

$$\chi^{\alpha_1 \dots \alpha_N}_{\lambda_1 \dots \lambda_N}(x_1, \dots, x_N) = \sum_\sigma (-1)^\sigma \prod_{k=1}^N \chi^{\alpha_{\sigma_k}}_{\lambda_k}(x_{\sigma_k}). \tag{10}$$

The energy of the N-particle state is equal to

$$E_N(\lambda_1, \dots, \lambda_N) = \sum_{k=1}^N \epsilon(\lambda_k), \qquad \epsilon(\lambda) = m_0 \operatorname{ch} \lambda. \tag{11}$$

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The periodic boundary condition

$$\chi^{\alpha_1 \alpha_2 \dots \alpha_N}(x_1 + L, x_2, \dots, x_N) = \chi^{\alpha_1 \alpha_2 \dots \alpha_N}(x_1, x_2, \dots, x_N)$$

yields

$$e^{im_0 L \operatorname{sh} \lambda_k} = 1, \qquad k = 1, \dots, N. \tag{12}$$

Free fermion: vacuum state

Take the logarithm on the last equation

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$$\lambda_k = i\pi + \xi_k, \qquad \xi_k \in \mathbb{R}.$$

We have to cut the band from below:

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The vacuum energy in the thermodynamic limit $L \to \infty$:

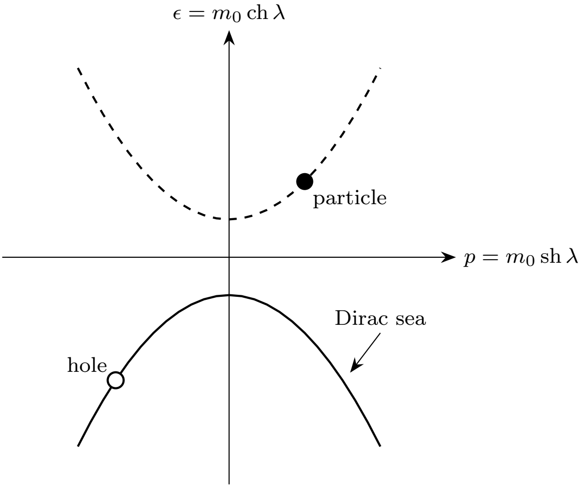
$$E_0 = -L\int_{-\Theta}^{\Theta} \frac{d\xi}{2\pi}\,\rho(\xi) m_0 \operatorname{ch}\xi, \qquad \rho(\xi) = \frac{2\pi}{L}\left|\frac{dn}{d\xi}\right| = m_0 \operatorname{ch}\xi.$$

Here $\rho(\xi)$ is the spectral density of states of the valence band.

Free fermion: excitations

There are two types of excitations:

- Particles: $\lambda_k \in \mathbb{R}$;
- Antiparticles or holes: absence of some $\lambda_k = i\pi + \xi_k$.



Interaction: Hamiltonian and wave function discontinuity

For $g \neq 0$ we have

$$\hat{H}_N = \sum_{k=1}^{N} (-i\sigma_k^3 \partial_{x_k} + m_0 \sigma_k^2) + g \sum_{k<l}^{N} \delta(x_k - x_l)(1 - \sigma_k^3 \sigma_l^3). \tag{14}$$

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Here

$$\frac{1}{2}(1 \otimes 1 - \sigma^3 \otimes \sigma^3)_{\alpha_1 \alpha_2}^{\alpha_1' \alpha_2'} = \delta_{\alpha_1}^{\alpha_1'} \delta_{\alpha_2}^{\alpha_2'} \delta_{\alpha_1, -\alpha_2}. \tag{15}$$

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Let us regularize the delta-function in an arbitrary way:

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$$\frac{f'(x)}{f(x)} = c\epsilon_a'(x) + \cancel{\frac{g(x, f(x))}{f(x)}} \quad \Rightarrow \quad f(x) = \text{const}\, e^{c\epsilon_a(x)}.$$

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$$f(+a) = e^c f(-a) \quad \xrightarrow{a\to 0} \quad f(+0) = e^c f(-0). \qquad (17)$$

Interaction: two-particle wave function

Let

$$\chi^{\alpha_1\alpha_2}_{\lambda_1\lambda_2}(x_1,x_2) = \begin{cases} A_{12}\chi^{\alpha_1}_{\lambda_1}(x_1)\chi^{\alpha_2}_{\lambda_2}(x_2) - A_{21}\chi^{\alpha_1}_{\lambda_2}(x_1)\chi^{\alpha_2}_{\lambda_1}(x_2) & \text{for } x_1 < x_2, \\ A_{21}\chi^{\alpha_1}_{\lambda_1}(x_1)\chi^{\alpha_2}_{\lambda_2}(x_2) - A_{12}\chi^{\alpha_1}_{\lambda_2}(x_1)\chi^{\alpha_2}_{\lambda_1}(x_2) & \text{for } x_1 > x_2. \end{cases} \tag{18}$$

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$$\frac{A_{21}}{A_{12}} = R(\lambda_1-\lambda_2), \qquad R(\lambda) = e^{i\Phi(\lambda)} = \frac{\operatorname{ch}\frac{\lambda-ig}{2}}{\operatorname{ch}\frac{\lambda+ig}{2}}. \tag{19}$$

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For $\Phi(\lambda)$ assume the skew symmetry

$$\Phi(-\lambda) = -\Phi(\lambda), \tag{21}$$

with the cuts lie on the rays $(i(\pi-|g|), i\infty)$, $(-i(\pi-|g|), -i\infty)$.

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Interaction: Bethe Ansatz and Bethe equations

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imposes on λ_k the system of Bethe equations:

$$e^{im_0 L \operatorname{sh}\lambda_k} \prod_{\substack{l=1\\ l\neq k}}^{N} R(\lambda_k - \lambda_l) = 1. \tag{24}$$

The N-particle solution (Bethe Ansatz):

$$\chi^{\alpha_1\ldots\alpha_N}_{\lambda_1\ldots\lambda_N}(x_1,\ldots,x_N)=\sum_{\tau}(-1)^{\sigma\tau}A_\tau\prod_{k=1}^{N}\chi^{\alpha_{\sigma_k}}_{\lambda_{\tau_k}}(x_{\sigma_k})\quad\text{for } x_{\sigma_1}<\ldots<x_{\sigma_N}. \tag{22}$$

The coefficients A satisfy the relations

$$A_{\ldots,i+1,i,\ldots}=R(\lambda_i-\lambda_{i+1})A_{\ldots,i,i+1,\ldots}. \tag{23}$$

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A set $\{\lambda_1,\ldots,\lambda_N\}$ that satisfy (24) is called a solution to the Bethe equations, while each element is called a root of the Bethe equations. All roots are different:

$$\lambda_k\neq\lambda_l,\quad \text{if } k\neq l. \tag{25}$$

Take logarithm of the Bethe equations:

$$m_0 L \,\mathrm{sh}\, \lambda_k + \sum_{l=1}^{N} \Phi(\lambda_k - \lambda_l) = 2\pi n_k, \qquad n_k \in \mathbb{Z}. \tag{26}$$

Ground state

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Energy and momentum:

$$E_N(\lambda_1, \dots, \lambda_N) = m_0 \sum_{i=1}^{N} \mathrm{ch}\, \lambda_i, \qquad P_N(\lambda_1, \dots, \lambda_N) = m_0 \sum_{i=1}^{N} \mathrm{sh}\, \lambda_i. \tag{27}$$

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$$\lambda_k = i\pi + \xi_k, \qquad n_k = k_0 - k.$$

Hence,

$$m_0 L \,\mathrm{sh}\,\xi_k = 2\pi(k - k_0) + \sum_{l=1}^{N} \Phi(\xi_k - \xi_l). \tag{28}$$

Ground state: thermodynamic limit

Take the difference for neighboring k and divide by $L(\xi_{k+1} - \xi_k)$:

$$m_0 \frac{\operatorname{sh} \xi_{k+1} - \operatorname{sh} \xi_k}{\xi_{k+1} - \xi_k} = \frac{2\pi}{L(\xi_{k+1} - \xi_k)} + \frac{1}{L} \sum_{l=1}^{N} \frac{\Phi(\xi_{k+1} - \xi_l) - \Phi(\xi_k - \xi_l)}{\xi_{k+1} - \xi_k}.$$

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In the limit $L \to \infty$ the values ξ_k become dense. Then define

$$\rho(\xi) = \frac{2\pi}{L(\xi_{k+1} - \xi_k)} = \frac{2\pi}{L} \frac{dk}{d\xi}. \tag{29}$$

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$$m_0 \operatorname{ch}\xi = \rho(\xi) + \int_{-\Theta}^{\Theta} \frac{d\xi\,'}{2\pi}\,\Phi'(\xi-\xi')\rho(\xi'). \tag{30}$$

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Approximate solution

$$\rho(\xi)=\rho_0\operatorname{ch}\frac{\pi\xi}{\pi+g},\qquad \rho_0\sim m_0e^{\frac{g\Theta}{\pi+g}}. \tag{32}$$

ρ_0 is finite $\Rightarrow$ renormalization of m_0.

Excitations in the thermodynamic limit: holes

Consider the Dirac sea with holes: n_k do not cover a segment of $\mathbb{Z}$. Then

$$m_0 L \operatorname{sh} \xi_k = -2\pi n_k + \sum_{l=1}^{N} \Phi(\xi_k - \xi_l). \tag{33}$$

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- Density of states: $\rho(\xi(n)) = \frac{2\pi}{L|\xi(n+1)-\xi(n)|} \simeq \frac{2\pi}{L}\left|\frac{dn}{d\xi(n)}\right|$.

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- Density of particles $\rho^\bullet(\xi) = \left\langle \frac{2\pi}{L|\xi_{k+1}-\xi_k|} \right\rangle_{\xi_k \simeq \xi} = \left\langle \frac{2\pi}{L}\left|\frac{dk}{d\xi_k}\right| \right\rangle_{\xi_k \simeq \xi}$.

There difference $\rho(\xi) - \rho^\bullet(\xi) = \rho^\circ(\xi)$ is the density of holes.

Integral equation for holes

Integral equation

$$m_0 \operatorname{ch} \xi = \rho(\xi) + \int_{-\Theta}^{\Theta} \frac{d\xi'}{2\pi} \, \Phi'(\xi - \xi')(\rho(\xi') - \rho^\circ(\xi')). \tag{35}$$

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Take the difference of (35) and (30):

$$\delta\rho(\xi) + \int_{-\Theta}^{\Theta} \frac{d\xi'}{2\pi} \Phi'(\xi - \xi')\delta\rho(\xi') = \int_{-\Theta}^{\Theta} \frac{d\xi'}{2\pi} \Phi'(\xi - \xi')\rho^{\circ}(\xi'). \tag{36}$$

Let us solve it in the limit $\Theta \to \infty$.

The limit $\Theta \to \infty$. Fourier transform

For $\Theta \to \infty$ we may apply the Fourier transform:

$$\tilde{X}(\omega) = \int_{-\infty}^{\infty} \frac{d\xi}{2\pi} X(\xi) e^{i\xi\omega}, \quad X = \Phi', \delta\rho, \rho^\circ, \ldots$$

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It is easy to check that

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Fractional charge of a hole

Effect of a cutoff: fractional charge. Formal charge of a hole is -1, but when holes are inserted, particles are pulled into the region $-\Theta \le \xi \le \Theta$ or pushed off it. We have two quantities:

$$\Delta N = -L \int_{-\Theta}^{\Theta} \frac{d\xi}{2\pi}\, \rho^{\circ}(\xi) = -\tilde{\rho}^{\circ}(0), \quad \Delta Q = L \int_{-\Theta}^{\Theta} \frac{d\xi}{2\pi}\, (\delta\rho(\xi) - \rho^{\circ}(\xi)). \tag{38}$$

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$$Q = |z^\circ|\int dx\,\psi^+_{\text{phys}}\psi_{\text{phys}} + \text{const} \quad \Rightarrow \quad \psi = |z^\circ|^{1/2}\psi_{\text{phys}}. \tag{41}$$

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It corresponds to the renormalization of the coupling constant g:

$$g_{\rm phys} = g|z^\circ| = \frac{g}{1-g/\pi} \quad\Leftrightarrow\quad \frac{1}{g_{\rm phys}} = \frac{1}{g}-\frac{1}{\pi}. \tag{42}$$

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$$-\pi < g < \pi \quad \Leftrightarrow \quad -\frac{\pi}{2} < g_{\text{phys}} < \infty$$

in consistency with the boson–fermion correspondence.

Mass of a hole

Calculate the energy and momentum of a state with holes:

$$E[\rho^\circ] - E[0] = m_0 L \int_{-\Theta}^{\Theta} \frac{d\xi}{2\pi} \left(\rho^\circ(\xi) - \delta\rho(\xi)\right) \operatorname{ch} \xi,$$

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Leg $g > 0$. The integrals diverge as $\Theta \to \infty$. Thus m_0 must be renormalized to zero.
Assuming Θ large but finite, we can obtain

$$E[\rho^\circ] - E[0] = L \int_{-\infty}^{\infty} \frac{d\xi}{2\pi}\, \epsilon(\xi)\rho^\circ(\xi), \qquad P[\rho^\circ] = L \int_{-\infty}^{\infty} \frac{d\xi}{2\pi}\, p(\xi)\rho^\circ(\xi) \tag{43}$$

where

$$\epsilon(\lambda) = m \operatorname{ch} \frac{\pi\lambda}{\pi + g}, \qquad p(\lambda) = m \operatorname{sh} \frac{\pi\lambda}{\pi + g}, \qquad m = \frac{M}{g} \operatorname{ctg}\left(\frac{\pi}{2} \frac{\pi - g}{\pi + g}\right) \tag{44}$$

and

$$m_0 = M \exp\left(-\frac{g}{\pi + g}\Theta\right) \sim M \left(\frac{m_0}{\Lambda}\right)^{g/(\pi+g)}. \tag{45}$$

Thus, the particles have a relativistic spectrum with the mass

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$$m_0 \sim m^{2-\beta^2},$$

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In terms of the physical coupling constant g_{phys} it reads

$$\frac{g_{\text{phys}}}{\pi} = \beta^{-2} - 1,$$

in consistency with the bosonization.

Model system

Consider the model system of 'spinless' fermions with the S matrix $S(\theta) = e^{i\Psi(\theta)}$, $\Psi(-\theta) = -\Psi(\theta)$. If particles are far from each other we may apply the Bethe Ansatz to them and obtain the Bethe equations

$$e^{imL\,\mathrm{sh}\,\theta_k} \prod_{\substack{l=1 \\ l\neq k}}^{N} S(\theta_l - \theta_k) = 1, \tag{46}$$

where θ_k are physical rapidities of the particles and m is their physical mass.

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Introduce the densities:

$$\begin{aligned} \rho_*(\theta(n)) &= \frac{2\pi}{L|\theta(n+1) - \theta(n)|} \simeq \frac{2\pi}{L}\left|\frac{dn}{d\theta(n)}\right|, \\ \rho_*^\bullet(\theta) &= \left\langle \frac{2\pi}{L|\theta_{k+1} - \theta_k|} \right\rangle_{\theta_k \simeq \theta} = \left\langle \frac{2\pi}{L}\left|\frac{dk}{d\theta_k}\right| \right\rangle_{\theta_k \simeq \theta}. \end{aligned} \tag{49}$$

Model system: integral equations

In the same way we obtain the integral equation

$$m \operatorname{ch} \theta + \int_{-\infty}^{\infty} \frac{d\theta'}{2\pi} \Psi'(\theta - \theta') \rho_*^{\bullet}(\theta') = 2\pi \rho_*(\theta). \tag{50}$$

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$$\delta\tilde{\rho}_*(\omega) = \tilde{\Psi}'(\omega)\rho_*^\bullet(\omega). \tag{52}$$

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$$\tilde{\rho}^\circ\left(\frac{\pi}{\pi+g}\omega\right) = \tilde{\rho}_*^\bullet(\omega), \qquad \delta\tilde{\rho}\left(\frac{\pi}{\pi+g}\omega\right) = \delta\tilde{\rho}_*(\omega). \tag{54}$$

Comparing it with (37) we obtain

$$\Psi(\theta) = i\int_{-\infty}^{\infty} \frac{d\omega}{\omega} \frac{\operatorname{sh}\frac{\pi\omega}{2}\operatorname{sh}\frac{\pi(p-1)\omega}{2}}{\operatorname{sh}\pi\omega\operatorname{sh}\frac{\pi p\omega}{2}} e^{-i\theta\omega}$$
$$= 2\int_{0}^{\infty} \frac{d\omega}{\omega} \frac{\operatorname{sh}\frac{\pi\omega}{2}\operatorname{sh}\frac{\pi(p-1)\omega}{2}}{\operatorname{sh}\pi\omega\operatorname{sh}\frac{\pi p\omega}{2}} \sin\theta\omega, \quad \beta^2 = 1 - \frac{g}{\pi} = 2\frac{p}{p+1}. \tag{55}$$

This identifies the function $S(\theta) = e^{i\Psi(\theta)}$ with the matrix element $S(\theta)^{--}_{--}$ of two antifermions in the Thirring model.

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This identifies the function $S(\theta) = e^{i\Psi(\theta)}$ with the matrix element $S(\theta)^{--}_{--}$ of two antifermions in the Thirring model.
Other matrix elements can be found by means of the Yang–Baxter equations and in the basis $(++, +-, -+, --)$ are given by

$$S(\theta) = \left(S^{\alpha_1\alpha_2}_{\beta_1\beta_2}(\theta)\right) = \begin{pmatrix} a(\theta) & & & \\ & b(\theta) & c(\theta) & \\ & c(\theta) & b(\theta) & \\ & & & a(\theta) \end{pmatrix}, \tag{56}$$

where $a(\theta) = e^{i\Psi(\theta)}$, and

$$\frac{b(\theta)}{a(\theta)} = \frac{\operatorname{sh}\frac{\theta}{p}}{\operatorname{sh}\frac{i\pi-\theta}{p}}, \qquad \frac{c(\theta)}{a(\theta)} = \frac{\operatorname{sh}\frac{i\pi}{p}}{\operatorname{sh}\frac{i\pi-\theta}{p}}. \tag{57}$$

For $g < 0$ (or $g_{\text{phys}} < 0$, $\beta^2 < 1$, $p < 1$) the elements $b(\theta)$ and $c(\theta)$ have poles on the physical sheet at the points

$$\theta_n = i\pi - i\pi p n, \qquad n = 1, 2, \ldots, \left\lfloor \frac{1}{p} \right\rfloor, \tag{58}$$

which correspond to the neutral bound states with the masses

$$M_n = 2m \sin \frac{\pi p n}{2} \tag{59}$$

In the sine-Gordon model these bound states correspond to the breather excitations, and in the classical limit $\beta^2 \to 0$ their spectrum becomes continuous in consistency with the classical field theory.