Lecture 6
$O(N)$-model: integrability and the exact S-matrix

Consider the $O(N)$-model with the action

$$S[\boldsymbol{n},\omega]=\frac{1}{2g}\int d^2x\,((\partial_\mu\boldsymbol{n})^2-\omega(\boldsymbol{n}^2-1))$$

and the classical equations of motion:

$$\partial^\mu\partial_\mu\boldsymbol{n}+\omega\boldsymbol{n}=0,\qquad \boldsymbol{n}^2=1.$$

In the light cone coordinates z, $\bar z$ we have for the action

$$S[\boldsymbol{n},\omega]=-\frac{1}{g}\int dz\,d\bar z\left(\partial\boldsymbol{n}\bar\partial\boldsymbol{n}+\frac{\omega}{4}(\boldsymbol{n}^2-1)\right),\tag{1}$$

and for the equations of motion we have

$$4\partial\bar\partial\boldsymbol{n}=\omega\boldsymbol{n},\qquad \boldsymbol{n}^2=1.\tag{2}$$

The action (1) is invariant with respect to *pseudoconformal* transformations

$$\boldsymbol{n}(z,\bar z)\to\boldsymbol{n}(f_1(z),f_2(\bar z)),\qquad \omega(z,\bar z)\to f_1'(z)f_2'(\bar z)\omega(f_1(z),f_2(\bar z)).\tag{3}$$

The transformations include, in particular, translations

$$f_1(z)=z+c,\qquad f_2(\bar z)=\bar z+\bar c,$$

scaling and Lorentz transformations, for which

$$f_1(z)=\lambda z,\qquad f_2(\bar z)=\bar\lambda\bar z,$$

and the inversion transformation

$$f_1(z)=1/z,\qquad f_2(\bar z)=1/\bar z.$$

In the Minkowski space the parameters with and without a bar are real and unrelated, while in the Euclidean space they are complex and complex conjugate.

Upon transition to the Euclidean space the translations, the scale transformation and inversion form a global conformal group consisting of conformal transformations, which are one-to-one transformations on the sphere $\mathbb{C}\cup\{\infty\}$. Local conformal transformations, that is, transformations that are one-to-one only on certain domains, are given in this case by arbitrary analytic functions $f(z)\equiv f_1(z)=\overline{f_2(\bar z)}$.

In the meantime, we continue consideration in the Minkowski space. The energy-momentum tensor has the form

$$T_{zz}=\frac{1}{g}(\partial\boldsymbol{n})^2,\qquad T_{\bar z\bar z}=\frac{1}{g}(\bar\partial\boldsymbol{n})^2,\qquad T_{z\bar z}=T_{\bar zz}=-\frac{\omega}{4g}(\boldsymbol{n}^2-1).$$

On the equations of motion the component $T_{z\bar z}=T_{\bar zz}$ vanishes, that is, $T^\mu_\mu=0$, which expresses the scale invariance of the model. The energy-momentum conservation is written as

$$\bar\partial(\partial\boldsymbol{n})^2=0,\qquad \partial(\bar\partial\boldsymbol{n})^2=0.\tag{4}$$

In most cases, energy and momentum are the only local integrals of motion, but in the case of the $O(N)$-model this is not so. First, there exists a trivial integral of motion in the form of a square of the energy-momentum tensor:

$$\bar\partial(\partial\boldsymbol{n})^4=0,\qquad \partial(\bar\partial\boldsymbol{n})^4=0.\tag{5}$$

Second, it is easy to obtain the relation

$$4\bar\partial(\partial^2\boldsymbol{n})^2=\partial(\omega(\partial\boldsymbol{n})^2)-3\partial\omega(\partial\boldsymbol{n})^2,\qquad 4\partial(\bar\partial^2\boldsymbol{n})^2=\bar\partial(\omega(\bar\partial\boldsymbol{n})^2)-3\bar\partial\omega(\bar\partial\boldsymbol{n})^2.\tag{6}$$

Now we will show that this equation means the existence of an additional integral of motion. Since $(\partial \boldsymbol{n})^2$ is conserved, by using the pseudoconformal transformation $z = f_1(z')$ such that

$$dz' = \left| \frac{\partial \boldsymbol{n}}{\partial z} \right| dz,$$

it is possible to achieve that in the new coordinates

$$(\partial \boldsymbol{n})^2 = 1.$$

Then (6) reduces to the form of the continuity equation

$$\bar{\partial}(\partial\partial\boldsymbol{n})^2 = \partial(2\partial\boldsymbol{n}\bar{\partial}\boldsymbol{n}).$$

In fact (although this is not easy to show), the model has an infinite number of integrals of motion in involution.

We know that in the quantum case the (pseudo) conformal invariance of the model is violated, since the field ω acquires a nonzero average. Therefore, the arguments with conformal transformations lose their power. In the quantum case in the equations, from which the conservation laws follow, anomalies should arise. The anomalous terms should not violate the scale invariance of the equations, into which conservation laws turn. Therefore, for example, in the right-hand side of the energy-momentum conservation law, only one anomalous term is allowed, which reduces to a total derivative

$$\bar{\partial}(\partial\boldsymbol{n})^2 = -\beta\partial\omega. \tag{7}$$

Analogously the relation (5) is modified. The r.h.s. is not already a total derivative:

$$\bar{\partial}(\partial\boldsymbol{n})^4 = -(2\beta + \alpha')(\partial\boldsymbol{n})^2\partial\omega + \partial(\ldots). \tag{8}$$

The same for (6):

$$4\bar{\partial}(\partial^2\boldsymbol{n})^2 = -(3 + \alpha)(\partial\boldsymbol{n})^2\partial\omega + \partial(\ldots). \tag{9}$$

From these three equations, one conservation law can be compiled:

$$\bar{\partial}\left(4(\partial^2\boldsymbol{n})^2 - \frac{3+\alpha}{2\beta+\alpha'}(\partial\boldsymbol{n})^4\right) = \partial(\ldots). \tag{10}$$

So, there are at least two integrals of motion of spin 1 and spin 3:

$$I_1 = \int dz\,(\partial\boldsymbol{n})^2, \qquad I_3 = \int dz\left(4(\partial^2\boldsymbol{n})^2 - \frac{3+\alpha}{2\beta+\alpha'}(\partial\boldsymbol{n})^4\right). \tag{11}$$

These integrals satisfy the equation $\bar{\partial}I_s = 0$. The integrals I_{-1}, I_{-3}, which satisfy the equation$\partial I_{-s} = 0$ can be obtained by the substitution $z \leftrightarrow \bar{z}$. When passing to the usual coordinates x, t, both of these quantities remain integrals of motion.

It is possible to show that these integrals commute. Fro the existence of four integrals of motion one can conclude that multiple production of particles in a collision of two particles is impossible.

Let $|\theta_1, \ldots, \theta_n\rangle$ be an asymptotic state of n particles with rapidities $\theta_1, \ldots, \theta_n$. Using the fact that $p_z = -\frac{m}{2}e^{\theta}$, $p_{\bar{z}} = \frac{m}{2}e^{-\theta}$, it is easy to check that

$$I_{\pm1}|\theta_1, \ldots, \theta_n\rangle = \mathrm{const}\sum_{i=1}^{n} me^{\pm\theta_i}|\theta_1, \ldots, \theta_n\rangle,$$
$$I_{\pm3}|\theta_1, \ldots, \theta_n\rangle = \mathrm{const}\sum_{i=1}^{n} m^3e^{\pm3\theta_i}|\theta_1, \ldots, \theta_n\rangle.$$

From this we get four equations for the scattering of two particles into n particles:

$$e^{s\theta_1} + e^{s\theta_2} = \sum_{i=1}^{n} e^{s\theta_i'} \qquad (s = -3, -1, 1, 3).$$

If we fix the particle rapidities in the final state, there will be four equations for two unknowns. These equations can have solutions only for special values of final rapidities $\theta'_1, \ldots, \theta'_n$. But from the analyticity of the amplitudes it follows that the amplitudes of such processes should be identically equal to zero. The only exception is the case $n = 2$, when the amplitudes can contain δ-functions corresponding to the boundary values of the poles off the mass shell.

It can be shown that the model contains an infinite number of integrals of motion I_s with odd spins s. In the general case, we have

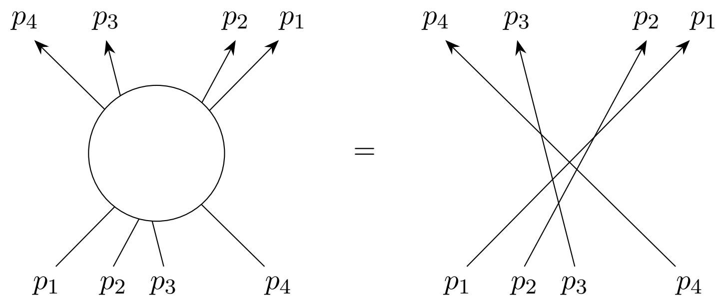
$$I_s|\theta_1, \ldots, \theta_n\rangle = \text{const} \sum_{i=1}^{n} e^{s\theta_i} |\theta_1, \ldots, \theta_n\rangle.$$

It follows that the model only admits scattering of n particles in n ones, and the particles can only exchange momenta.

Now do the important

Factorized scattering assumption. *The scattering amplitude of n particles into n particles factorizes into the product of all pairwise scattering amplitudes in any order with summation over the internal states of the intermediate particles.*

Graphically, this assumption can be represented as follows:



(12)

In principle, the factorized scattering conjecture can be checked in the diagram technique order by order in $1/N$. But one can use the following qualitative consideration. Suppose that there is a finite radius of interaction of particles R, beyond which virtual particles are almost not born. This means that if $|x_i - x_j| \gg R$ $(\forall i, j)$, the wave function almost indistinguishable from the wave function n free particles. Due to the existence of $I_{\pm 3}$, pairwise scattering of particles can be reduced to the passage of particles through each other with a change in internal states. Therefore, one can choose a basis of wave functions without any reflected waves. Let σ, τ be elements of the permutation group S_n of the numbers $1, \ldots, n$. Then the system of n bosons will be described by the wave function

$$\psi_{\beta_1 p_1, \ldots, \beta_n p_n}(\alpha_1 x_1, \ldots, \alpha_n x_n) = \sum_{\tau \in S_n} A^{\alpha_{\sigma_1} \ldots \alpha_{\sigma_n}}_{\beta_1 \quad \ldots \beta_n}[\tau] e^{i \sum_{i=1}^{n} p_{\tau_i} x_{\sigma_i}}$$

$$\text{for} \quad x_{\sigma_1} < x_{\sigma_2} < \cdots < x_{\sigma_n}, \quad |x_i - x_j| \gg R. \quad (13)$$

We have omitted the dependence of the coefficients A on the momenta. It is easy to verify that the function (13) is symmetric with respect to the permutations of the pairs $\alpha_i x_i \leftrightarrow \alpha_j x_j$.

We have not yet defined the meaning of the parameters β_i. In principle, we may not do this. But if we want β_i to match, say, the state α_i of the incoming particle i, we can require

$$A^{\alpha_1 \ldots \alpha_n}_{\beta_1 \ldots \beta_n}[\mathrm{id}] = \prod_{i=1}^{n} \delta^{\alpha_i}_{\beta_i}.$$

For $p_1 > p_2 > \cdots > p_n$, the parameters β_i naturally describe the internal states of the incoming particles. Although with this definition we lose symmetry with respect to permutations of the pairs $\beta_i p_i \leftrightarrow \beta_j p_j$, but the functions become analytic in the momenta.

The permutation of two particles is equivalent to the scattering of these particles. Of course, scattering changes the actual states of the particles α_i, and not the labels of the wave function β_i. Let $s^i \in S_n$ be a permutation of the numbers i and $i+1$, that is, $s^i_i = i+1$, $s^i_{i+1} = i$, $s^i_j = j$ $(j \neq i, i+1)$. Then

$$A^{\alpha_1 \ldots \alpha_{i+1} \alpha_i \ldots \alpha_n}_{\beta_1 \ldots \beta_i \, \beta_{i+1} \ldots \beta_n}[\tau s^i] = \sum_{\alpha'_i \alpha'_{i+1}} S^{\alpha_i \alpha_{i+1}}_{\alpha'_i \alpha'_{i+1}}(p_{\tau_i}, p_{\tau_{i+1}}) A^{\alpha_1 \ldots \alpha'_i \alpha'_{i+1} \ldots \alpha_n}_{\beta_1 \ldots \beta_i \, \beta_{i+1} \ldots \beta_n}[\tau]. \quad (14)$$

Now we permute three consecutive particles, for example, $123 \to 321$. Such a transition can be performed in two ways:

$$
\begin{tikzcd}[column sep=small, row sep=small]
 & 132 \arrow[r] & 312 \arrow[dr] & \\
123 \arrow[ur] \arrow[dr] & & & 321. \\
 & 213 \arrow[r] & 231 \arrow[ur] &
\end{tikzcd}
$$

The first way leads to the relation

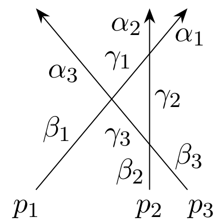
$$
A^{\alpha_3\alpha_2\alpha_1\cdots}_{\cdots}[321\ldots] = \sum_{\beta_1,\beta_2,\beta_3} \left(\sum_{\gamma_1,\gamma_2,\gamma_3} S^{\alpha_1\alpha_2}_{\gamma_1\gamma_2}(p_1,p_2) S^{\gamma_1\alpha_3}_{\beta_1\gamma_3}(p_1,p_3) S^{\gamma_2\gamma_3}_{\beta_2\beta_3}(p_2,p_3) \right) A^{\beta_1\beta_2\beta_3\cdots}_{\cdots}[123\ldots]. \tag{15}
$$

It is easier to write it down in the matrix form

$$
A_{321\ldots} = S_{12}(p_1,p_2) S_{13}(p_1,p_3) S_{23}(p_2,p_3) A_{123\ldots},
$$

where the $1,2,3$ subscripts indicate the number of the space, on which the matrices act or in which the vectors live. It is even more convenient to depict it graphically

$$\sum_{\text{internal lines}}$$



The second method leads to a different relation

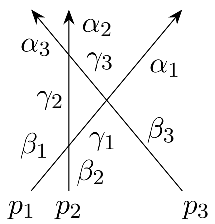
$$
A^{\alpha_3\alpha_2\alpha_1\cdots}_{\cdots}[321\ldots] = \sum_{\beta_1,\beta_2,\beta_3} \left(\sum_{\gamma_1,\gamma_2,\gamma_3} S^{\alpha_2\alpha_3}_{\gamma_2\gamma_3}(p_2,p_3) S^{\alpha_1\gamma_3}_{\gamma_1\beta_3}(p_1,p_3) S^{\gamma_1\gamma_2}_{\beta_1\beta_2}(p_1,p_2) \right) A^{\beta_1\beta_2\beta_3\cdots}_{\cdots}[123\ldots], \tag{16}
$$

or, simpler,

$$
A_{321\ldots} = S_{23}(p_2,p_3) S_{13}(p_1,p_3) S_{12}(p_1,p_2) A_{123\ldots},
$$

or, graphically,

$$\sum_{\text{internal lines}}$$



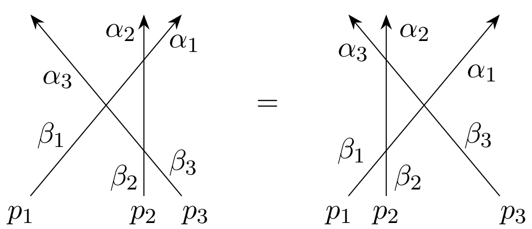
The condition that (15) and (16) lead to the same relation is called the *Yang–Baxter equation* and is written down as

$$
\sum_{\gamma_1,\gamma_2,\gamma_3} S^{\alpha_1\alpha_2}_{\gamma_1\gamma_2}(p_1,p_2) S^{\gamma_1\alpha_3}_{\beta_1\gamma_3}(p_1,p_3) S^{\gamma_2\gamma_3}_{\beta_2\beta_3}(p_2,p_3) = \sum_{\gamma_1,\gamma_2,\gamma_3} S^{\alpha_2\alpha_3}_{\gamma_2\gamma_3}(p_2,p_3) S^{\alpha_1\gamma_3}_{\gamma_1\beta_3}(p_1,p_3) S^{\gamma_1\gamma_2}_{\beta_1\beta_2}(p_1,p_2), \tag{17}
$$

or, shorter,

$$
S_{12}(p_1,p_2) S_{13}(p_1,p_3) S_{23}(p_2,p_3) = S_{23}(p_2,p_3) S_{13}(p_1,p_3) S_{12}(p_1,p_2), \tag{18}
$$

or, graphically,



(19)

Two graphs on (19) differ in the position of one of the lines (for example, the second). In the first graph, it lies to the left of the vertex, where the first and third particles intersect, and in the second, to the right of it. In other words, the Yang–Baxter equation expresses the condition that the lines in (12) can be shifted as you like, passing through the vertices. That is, it does not matter in which order we consider pairwise scattering of particles: in any case we will get the same answer.

The second condition on the S-matrix is more elementary. Let us return to the relation (14). It is clear that if we twice permute two consecutive indices in the coefficients A, i.e., perform the transformations

$$12 \to 21 \to 12,$$

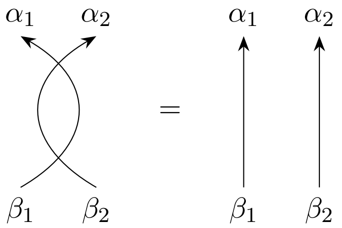
we must get an identical transformation. This implies the *unitarity condition*

$$\sum_{\gamma_1,\gamma_2} S^{\alpha_1\alpha_2}_{\gamma_1\gamma_2}(p_1,p_2) S^{\gamma_2\gamma_1}_{\beta_2\beta_1}(p_2,p_1) = \delta^{\alpha_1}_{\beta_1}\delta^{\alpha_2}_{\beta_2}, \tag{20}$$

or

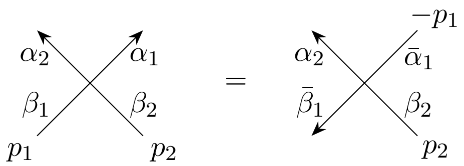
$$S_{12}(p_1,p_2)S_{21}(p_2,p_1) = 1, \tag{21}$$

or



(22)

The last *crossing invariance* condition is only true in a relativistic theory. It is natural to immediately write it graphically:



(23)

Here the momenta p_1 and p_2 are understood as spatially-temporal momenta, the bar over the index of the internal state of a particle represents the antiparticle. Formally it is written like this

$$S^{\alpha_1\alpha_2}_{\beta_1\beta_2}(p_1,p_2) = \sum_{\alpha_1'\beta_1'} C_{\beta_1\beta_1'} S^{\alpha_2\beta_1'}_{\beta_2\alpha_1'}(p_2,-p_1) C_{\alpha_1'\alpha_1}, \tag{24}$$

where C is the CPT conjugation matrix.

If we express the momenta in terms of rapidities, we obtain:

1. Yang–Baxter equation

$$S_{12}(\theta_1-\theta_2)S_{13}(\theta_1-\theta_3)S_{23}(\theta_2-\theta_3) = S_{23}(\theta_2-\theta_3)S_{13}(\theta_1-\theta_3)S_{12}(\theta_1-\theta_2). \tag{25}$$

2. Unitarity

$$S_{12}(\theta)S_{21}(-\theta) = 1. \tag{26}$$

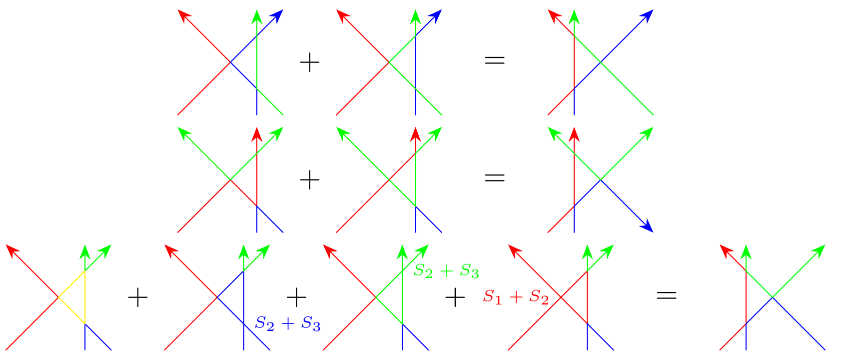
3. Crossing symmetry

$$S_{12}(\theta) = C_1 S_{2\tilde{1}}(i\pi-\theta)C_1, \tag{27}$$

where the tilde above the digit 1 means the transposition in the indices corresponding to this space.

The *bootstrap* conditions (25–27) are extremely restrictive. Together with the symmetry of the model and analyticity conditions, they make it possible to find an exact expression for the S-matrix. Consider the analyticity conditions. The S-matrix is a meromorphic function of θ. The area that corresponds to the physical sheet is

$$0 \le \operatorname{Im}\theta < \pi, \tag{28}$$



(31)

(32)

(33)

Figure 1: Equations (31)–(33) depicted schematically. Connection of two lines of the same color at a vertex corresponds a Kronecker symbol in the S-matrix. On intersections of two lines of the same color we mark the corresponding contribution to the S-matrix.

and the point $\theta = i\pi$ corresponds to the branch point $s = (m_1 - m_2)^2$ in the variable

$$s = m_1^2 + m_2^2 + 2m_1 m_2 \operatorname{ch}\theta,$$

and the point $\theta = 0$ corresponds to the point $s = (m_1 + m_2)^2$. The line $\operatorname{Im}\theta = \pi$ corresponds in the s-plane to the left cut $(-\infty, (m_1 - m_2)^2]$, while the line $\operatorname{Im}\theta = 0$ corresponds to the right cut $[(m_1 + m_2)^2, \infty)$.

On the imaginary axis the S-matrix is real:

$$S(iu) \in \mathbb{R} \qquad \text{for} \quad u \in \mathbb{R}, \tag{29}$$

and all the poles of the S-matrix on the physical sheet are situated on the imaginary axis. Some of these poles correspond to bound states, however, additional study is usually required to determine whether a given pole corresponds to a bound state.

Let us solve the Young–Baxter equation for $O(N)$-symmetric ($N \geq 3$) S-matrix of the size $N^2 \times N^2$ of the form [1]

$$S_{ij}^{i'j'}(\theta) = \delta_{i'j'}\delta_{ij}S_1(\theta) + \delta_{i'i}\delta_{j'j}S_2(\theta) + \delta_{j'i}\delta_{i'j}S_3(\theta). \tag{30}$$

The Yang–Baxter equation for it takes the form (Fig. 1)

$$S_2(\theta)S_3(\theta+\theta')S_3(\theta') + S_3(\theta)S_3(\theta+\theta')S_2(\theta') = S_3(\theta)S_2(\theta+\theta')S_3(\theta'), \tag{31}$$

$$S_2(\theta)S_1(\theta+\theta')S_1(\theta') + S_3(\theta)S_2(\theta+\theta')S_1(\theta') = S_3(\theta)S_1(\theta+\theta')S_2(\theta'), \tag{32}$$

$$\begin{aligned} NS_1(\theta)S_3(\theta+\theta')S_1(\theta') + S_1(\theta)S_3(\theta+\theta')(S_2(\theta') + S_3(\theta')) + (S_2(\theta) + S_3(\theta))S_3(\theta+\theta')S_1(\theta') \\ + S_1(\theta)(S_1(\theta+\theta') + S_2(\theta+\theta'))S_1(\theta') = S_3(\theta)S_1(\theta+\theta')S_3(\theta'). \end{aligned} \tag{33}$$

To solve this system introduce the notation $h(\theta) = S_2(\theta)/S_3(\theta)$. The first equation reads

$$h(\theta) + h(\theta') = h(\theta + \theta').$$

Therefore, $h(\theta) \sim \theta$ and

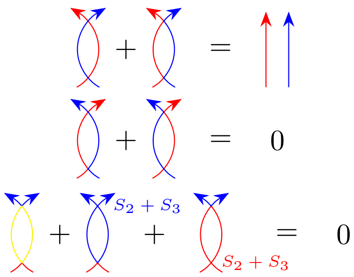
$$S_3(\theta) = -i\frac{\lambda}{\theta}S_2(\theta). \tag{34}$$

Now let $g(\theta) = S_2(\theta)/S_1(\theta)$. Substituting (34) into (32), we obtain

$$g(\theta + \theta') - g(\theta') = \frac{\theta}{i\lambda}.$$

This equation has a solution

$$g(\theta) = \frac{\theta - i\kappa}{i\lambda}.$$



(39)
(40)
(41)

Figure 2: Equations (39)–(41) depicted schematically.

Substituting it into (33), we get

$$\kappa = \frac{N-2}{2}\lambda.$$

It meas that

$$S_1(\theta) = -\frac{i\lambda}{i(N-2)\lambda/2 - \theta} S_2(\theta). \tag{35}$$

This is the most general solution to the Yang–Baxter equation, which depends on an arbitrary function $S_2(\theta)$ and an arbitrary parameter λ. Let us now use the conditions of crossing invariance and unitarity to fix S_2 and λ.

The crossing symmetry condition has the form

$$S_2(\theta) = S_2(i\pi - \theta), \tag{36}$$
$$S_1(\theta) = S_3(i\pi - \theta). \tag{37}$$

Substituting here (34) and (35), we obtain

$$\lambda = \frac{2\pi}{N-2}. \tag{38}$$

The unitarity condition (Fig. 2)

$$S_2(\theta)S_2(-\theta) + S_3(\theta)S_3(-\theta) = 1, \tag{39}$$
$$S_2(\theta)S_3(-\theta) + S_3(\theta)S_2(-\theta) = 0, \tag{40}$$
$$NS_1(\theta)S_1(-\theta) + S_1(\theta)(S_2(-\theta) + S_3(-\theta)) + (S_2(\theta) + S_3(\theta))S_1(-\theta) = 0 \tag{41}$$

$$S_2(\theta)S_2(-\theta) = \frac{\theta^2}{\theta^2 + \lambda^2}. \tag{42}$$

Now we have to solve the equations (36) and (42) together. It is clear that the solution to these equations is ambiguous. A solution turns into a solution, if you multiply it by a function

$$\frac{\operatorname{sh}\theta + i\sin\alpha}{\operatorname{sh}\theta - i\sin\alpha}$$

with arbitrary α. We will look for a "minimal" solution, that is, a solution that will have the least number of zeros and poles on the physical sheet.

From (42) we conclude that $S_2(\theta)$ has a simple zero at the point $\theta = 0$. From the crossing symmetry (36) we immediately conclude that a simple zero also exists at the point $\theta = i\pi$. From the unitarity we find that there is a pole at the point $\theta = -i\pi$. Continuing alternately applying the crossing symmetry and the unitarity, we find a set of poles and zeros of the function $S_2(\theta)$:

$$\begin{aligned} &\text{Zeros:} \quad && \theta = -2\pi i n, i\pi + 2\pi i n, \\ &\text{Poles:} \quad && \theta = -i\pi - 2\pi i n, 2\pi i + 2\pi i n, \qquad n = 0, 1, 2, \ldots \end{aligned} \tag{43}$$

Another set of zeros and poles is obtained as follows. From (42) it follows that S_2 must have a pole at one of the points $\theta = \mp i\lambda$. Let us denote solutions with such poles as $S_2^{(\pm)}(\theta)$. By reasoning as before, we obtain for $S_2^{(\pm)}(\theta)$

$$\begin{aligned} &\text{Zeros:} \quad \theta = \mp i\lambda - i\pi - 2\pi i n, \pm i\lambda + 2\pi i + 2\pi i n, \\ &\text{Poles:} \quad \theta = \mp i\lambda - 2\pi i n, \pm i\lambda + i\pi + 2\pi i n, \qquad n = 0, 1, 2, \ldots \end{aligned} \tag{44}$$

Gathering (43), (44), we obtain

$$S_2^{(\pm)}(\theta) = Q^{(\pm)}(\theta) Q^{(\pm)}(i\pi - \theta), \qquad Q^{(\pm)}(\theta) = \frac{\Gamma\left(\pm\frac{\lambda}{2\pi} - i\frac{\theta}{2\pi}\right)\Gamma\left(\frac{1}{2} - i\frac{\theta}{2\pi}\right)}{\Gamma\left(\frac{1}{2} \pm \frac{\lambda}{2\pi} - i\frac{\theta}{2\pi}\right)\Gamma\left(-i\frac{\theta}{2\pi}\right)}. \tag{45}$$

By expanding the S-matrix (34), (35), (45) with (38) in powers of $1/N$, we obtain

$$S_1^{(\pm)}(\theta) = -\frac{2\pi i}{N(i\pi - \theta)}, \tag{46}$$

$$S_2^{(\pm)}(\theta) = 1 \mp \frac{2\pi i}{N \operatorname{sh}\theta}, \tag{47}$$

$$S_3^{(\pm)}(\theta) = -\frac{2\pi i}{N\theta}. \tag{48}$$

This allows us to identify $S^{(+)}(\theta)$ with the S-matrix of the $O(N)$-model, and $S^{(-)}(\theta)$ with the S-matrix of the N-component Neveu–Schwartz model. Notice that $S_1^{(\pm)}(0) = S_2^{(\pm)}(0) = 0$, $S_3^{(\pm)}(0) = \mp 1$. It means that

$$S_{12}^{(\pm)}(0) = \mp P_{12}, \tag{49}$$

where $P_{12} : a \times b \mapsto b \times a$ is the permutation operator of the spaces 1 and 2. This means that for the particles in the $O(N)$-model a kind of the Pauli principle applies, although we considered the particles to be bosons. Two particles cannot have the same momentum.

In fact, in two-dimensional space-time it cannot be said whether the particle is a boson or a fermion. If we talk about the spin, then we do know what the spin of an operator is, but we do not know what the spin of a state is, since there are no rotations in a one-dimensional space. Besides, in one spatial dimension there is a way to construct a Clifford algebra (fermion algebra) from the Heisenberg algebra (boson algebra) and vice versa [2]. This transformation respects the concept of a particle, but changes the nature of the interaction. Namely, the scattering matrix of particles as fermions differs from the scattering matrix of the same particles as bosons by a sign.

Bibliography

[1] A. Zamolodchikov and Al. Zamolodchikov, *Annals of Physics* **120** (1979) 253.

[2] M. Sato, M. Jimbo, T. Miwa, Holonomic quantum fields I–V, a series of papers in Publications of RIMS, 1978–1980.

Problems

1. Obtain the equation (6).

2. Write explicitly the asymptotic expression for the wave function (13) for two and three particles ($n = 2, 3$). Having accepted the condition (14) for $n = 2$ for the definition of the S-matrix, make sure that this implies (14) for the $n = 3$ case. Show that the product of three S-matrices $S_{12}S_{13}S_{23}$ actually has the meaning of a three-particle S-matrix.

3. Derive (46–48).

4. Show that (49) implies the Pauli principle for *interacting boson* particles: two particles cannot have the same momentum.

5*. Show that the 4×4 matrix of the form

$$S(\theta)=\left(S^{\alpha_1\alpha_2}_{\beta_1\beta_2}(\theta)\right)=\begin{pmatrix} a(\theta) & & & \\ & b(\theta) & c(\theta) & \\ & c(\theta) & b(\theta) & \\ & & & a(\theta) \end{pmatrix}$$

(the indices $\alpha_1\alpha_2$ are ordered as $++,+-,-+,--$) satisfies the Yang–Baxter equation, if

$$\frac{b(\theta)}{a(\theta)}=\frac{\operatorname{sh}\frac{\theta}{p}}{\operatorname{sh}\frac{i\pi-\theta}{p}},\qquad \frac{c(\theta)}{a(\theta)}=\frac{\operatorname{sh}\frac{i\pi}{p}}{\operatorname{sh}\frac{i\pi-\theta}{p}}$$

for arbitrary p.

Such a S-matrix is a soliton scattering matrix in the sine-Gordon model with $\beta^2=2\frac{p}{p+1}$ for a suitable $a(\theta)$ such that $a(0)=-1$. Show that $a(\theta)$ should satisfy the conditions

$$a(\theta)=\frac{\operatorname{sh}\frac{i\pi-\theta}{p}}{\operatorname{sh}\frac{\theta}{p}}a(i\pi-\theta),\qquad a(\theta)a(-\theta)=1.$$