Lecture 3
Renormalization group for the Berezinskii–Kosterlitz–Thouless transition

We consider the BKT transition by using the sine-Gordon model. In this lecture, it will be more convenient for us to use the Euclidean representation:

$$S_{\rm SG}[\phi] = \int d^2x \left(\frac{(\partial_\mu \phi)^2}{8\pi} - \alpha_0 r_0^{\beta_0^2-2} \cos\beta_0\phi \right), \tag{1}$$

We put noughts at the constants in order to emphasize that these are non-renormalized constants. Later we will also use the renormalized constants α, β. In addition, in comparison with the action in the first lecture, we added the dimensionless constant α_0. In Lecture 2 we combined the constants α_0 and r_0 into the constant $\mu = \alpha_0 r_0^{\beta^2-2}$ and assumed that the correlation length $r_c \sim \mu^{-1/(2-\beta^2)}$ is much longer than the ultraviolet cutoff r_0. In the case of a relevant perturbation $\beta^2 < 2$, this allowed us to develop the perturbation theory with respect to the constant μ, since it corresponded to the case $\alpha \ll 1$. Here, on the contrary, we start with the case of $r_c \gtrsim r_0$, which is more natural for studying the BKT transition and see, how the theory will look as the scale grows. Generally speaking, this is not easy to do, but since we are interested in the neighborhood of the phase transition point $\beta^2 = 2 + \delta$, $|\delta| \ll 1$, we can use the renormalization-group approach along with the perturbation theory[1, 2].

There is also a purely field-theoretical interpretation of the renormalization group. Let us consider the system in the regime $\alpha_0 \ll 1$ ($r_c \gg r_0$), but we will consider correlation functions on the scales r such that $r_0 \ll r \ll r_c$. The correlation functions on the scale of r, calculated from the bare action taking with all the renormalizations taken into account, will behave as correlation functions in the tree approximation for the renormalized action with suitably renormalized coupling constants. If we know, how the renormalized coupling constants "run" with a scale of r, we will be able to calculate correlation functions in this intermediate region.

We said that in the unperturbed theory — the theory of a free massless field — there is a scale parameter R, which has the meaning of the size of the region in which the theory lives. It will be convenient for us to use it as a large-scale renormalization parameter, but it will be necessary to determine it in a slightly more accurate way. In fact, we could consider a theory on space with compact dimensions (compactification). For example, we could compactify the theory on a cylinder of a circle R, but this would violate isotropy. Alternative compactification onto the sphere would unduly complicate the calculations. Therefore, instead of compactifying the theory, we introduce a small mass term into the Lagrangian:

$$S_{SG}[\phi] = \int d^2x \left(\frac{(\partial_\mu \phi)^2}{8\pi} + \frac{m_0^2\phi^2}{8\pi} - \alpha_0 r_0^{\beta_0^2-2} \cos\beta_0\phi \right). \tag{2}$$

For ultraviolet regularization, we will replace x^2 by $x^2 + r_0^2$. Then for $m_0^2 x^2 \ll 1$ the free field propagator (with $\alpha_0 = 0$) is equal to

$$G_0(x - x') = \log \frac{R_0^2}{(x-x')^2 + r_0^2}, \qquad R_0 = (cm_0)^{-1}, \qquad c = e^{\gamma_E}/2. \tag{3}$$

Here γ_E is the Euler constant. We see that the inclusion of the mass term effectively changes correlation functions at small distances in the same way as compactification does.

Now write down the renormalized action

$$S^R_{\rm SG}[\phi] = \int d^2x \left(\frac{(\partial_\mu \phi)^2}{8\pi} + \frac{m^2\phi^2}{8\pi} - \frac{\alpha}{R^2} \cos\beta\phi \right), \qquad R = (cm)^{-1}, \tag{4}$$

such that $S_{\rm SG}[\phi] = S^R_{\rm SG}[Z_\phi^{-1/2}\phi] + S^{\rm ct}[Z_\phi^{-1/2}\phi]$. We will require that the counterterm contribution to the action $S^{\rm ct}[\phi]$ does not contain counterterms to the "mass" term:

$$S^{\rm ct}[\phi] = \int d^2x \, (\#(\partial_\mu\phi)^2 + \#\cos\beta\phi).$$

The presence of a counterterm to the kinetic term means that the field ϕ is renormalized. Although formally the letter ϕ in the action is an arbitrary function, we will consider correlation functions for both nonrenormalized and renormalized fields. To distinguish these two cases we will denote the renormalized field by ϕ_R. Besides, below we define the renormalized coupling constant α. Thus, the renormalization rule should look like:

$$\begin{aligned} \phi &= Z_\phi^{1/2}\phi_R, \\ \beta_0 &= Z_\phi^{-1/2}\beta, \\ \alpha_0 &= Z_\alpha \alpha, \\ m_0 &= Z_\phi^{-1/2} m, \\ R_0 &= Z_\phi^{1/2} R. \end{aligned} \tag{5}$$

Now consider the propagator $G_0(x-x') = \langle\phi(x)\phi(x')\rangle_0$ of the unperturbed ($\alpha_0 = 0$) theory and the propagator $G(x-x') = \langle\phi(x)\phi(x')\rangle$ of the complete theory for the nonrenormalized field. They can be considered as kernels of the integral operators $G_0 = 4\pi(-\partial_\mu^2 + m_0^2)^{-1}$ and G correspondingly. There is a relation between them

$$G^{-1} = G_0^{-1} + \frac{1}{4\pi}\Sigma, \tag{6}$$

where the operator Σ is called the mass operator. We normalized it so as to restore its natural meaning as corrections to the squared mass.

The renormalization group theory uses two approaches. One approach (Kadanov's approach) is used more often in statistical mechanics, and studies the evolution of seed coupling constants in the ultraviolet cutoff parameter $\Lambda \sim r_0^{-1}$ for given renormalized coupling constants. Another approach, commonly used in quantum field theory, assumes that the seed constants and the cutoff parameter are fixed (and thus define the theory). In this case, it turns out that there is some uncertainty in the definition of the renormalized coupling constants, depending on a dimensional parameter, for example, the renormalization point κ. The evolution of coupling constants in this dimensional parameter is studied. The advantage of the second method is that it can be used to extract equations for correlation functions by means of fairly direct methods. Therefore, we will take this approach. In our case, for such a dimensional parameter we can take the infrared cutoff R or, equivalently, the auxiliary mass m. Let us demand that in the vicinity of the point $p^2 = 0$ in the momentum space the renormalized propagator $G_R(p^2)$ has the natural form $4\pi(p^2 + M^2)^{-1}$, where M is a certain mass parameter. We use it to determine the renormalized coupling constant α. In the direct space we have

$$G_R(x-x') = Z_\phi^{-1} G(x-x') = \langle\phi_R(x)\phi_R(x')\rangle. \tag{7}$$

Our requirement is that in the momentum space

$$G_R(p^2) = \frac{4\pi}{p^2 + M^2} + O(p^4) \quad \text{as } p^2 \to 0, \tag{8}$$

with

$$M^2 = m^2 + \frac{4\pi\alpha\beta^2}{R^2} = m^2(1 + 4\pi c^2 \alpha\beta^2). \tag{9}$$

This condition can be rewritten as $\Sigma(p^2) = \Sigma_0 + \Sigma_1 p^2 + O(p^4)$. Indeed,

$$\begin{aligned} 4\pi G^{-1}(p^2) &= p^2 + m_0^2 + \Sigma(p^2) = p^2 + m_0^2 + \Sigma_0 + \Sigma_1 p^2 + O(p^4)) \\ &= (1+\Sigma_1)\left(p^2 + m^2 + \Sigma_0(1+\Sigma_1)^{-1}\right) + O(p^4) = 4\pi(1+\Sigma_1)G_R^{-1}(p^2). \end{aligned} \tag{10}$$

From this we derive

$$Z_\phi = \frac{1}{1+\Sigma_1}, \qquad M^2 = m^2 + \frac{\Sigma_0}{1+\Sigma_1}, \qquad m^2 = \frac{m_0^2}{1+\Sigma_1}. \tag{11}$$

It will be convenient for us to start calculating the mass operator by expanding for the propagator in the coordinate representation:

$$G(x-x') = \langle \phi(x)\phi(x')\rangle = \frac{\langle \phi(x)\phi(x')e^{-S_1[\phi]}\rangle_0}{\langle e^{-S_1[\phi]}\rangle_0}$$
$$= \langle \phi(x)\phi(x')\rangle_0 - \langle\phi(x)\phi(x')S_1[\phi]\rangle_{0,\mathrm{c}} + \frac{1}{2}\langle\phi(x)\phi(x')S_1^2[\phi]\rangle_{0,\mathrm{c}} - \frac{1}{6}\langle\phi(x)\phi(x')S_1^3[\phi]\rangle_{0,\mathrm{c}} + O(\alpha_0^4).$$

Here the denominator $\langle e^{-S_1}\rangle_0$ cancels the disconnected diagrams. To take this into account, only the connected parts of the correlation functions, designated here as $\langle\cdots\rangle_{0,\mathrm{c}}$, should be calculated. We will exclude disconnected diagrams on the fly. We will also exclude on the fly the one-particle-reducible diagrams and easily throw away the factors G_0 on "legs", so that, in fact, we will immediately calculate the mass operator.

To demonstrate the technique, we calculate the first order of the perturbation theory:

$$-\langle\phi(x)\phi(x')S_1[\phi]\rangle = \alpha_0 r_0^{\delta_0}\int d^2y\,\langle\phi(x)\phi(x')\,{:}\cos\beta_0\phi(y){:}\,\rangle_0.$$

The expectation value in the right hand side is easily calculated:

$$\langle\phi(x)\phi(x')\,{:}\cos\beta_0\phi(y){:}\,\rangle_0 = \langle\phi(x)\phi(x')\rangle_0\langle\,{:}\cos\beta_0\phi(y){:}\,\rangle_0 - \beta_0^2\langle\phi(x)\phi(y)\rangle_0\langle\phi(x')\phi(y)\rangle_0\langle\,{:}\cos\beta_0\phi(y){:}\,\rangle_0.$$

The first term is the sum of disconnected diagrams and is canceled with the corresponding contribution from $\langle e^{-S_1}\rangle_0$. The second term naturally splits into two "tail" lines and a contribution to the mass operator. Given that $\langle\,{:}\cos\beta_0\phi(y){:}\,\rangle_0 = R_0^{-\beta_0^2}$, for the first-order contribution to the mass operator we have

$$-\frac{1}{4\pi}\Sigma^{(1)}(y-y') = -\alpha_0\beta_0^2\frac{r_0^{\delta_0}}{R_0^{\beta_0^2}}\delta(y-y').$$

Therefore

$$\Sigma^{(1)}(p^2) = \Sigma_0^{(1)} = \frac{4\pi\alpha_0\beta_0^2}{R_0^2}\left(\frac{r_0}{R_0}\right)^{\delta_0}. \tag{12}$$

As expected, this formula is consistent with the semiclassical limit of $\beta\to 0$ ($\delta\to -2$). In this limit, the contribution of the infrared cutoff vanishes, and $M^2 = m_0^2 + 4\pi\alpha\beta^2 r_0^{-2} = m_0^2 + 4\pi\mu\beta^2$.

In the general case, we see that the mass depends on the parameter R of the infrared cutoff. Here we must take into account that the mass M is not the mass of elementary excitations in theory. The matter is that for $\beta^2 \le 2$ the theory is asymptotically free, that is, the field ϕ only describes excitations at small scales, while at large scales there is a completely different system of excitations, topological solitons and (for $\beta^2 < 1$) breathers. Here we will just be interested in the mass of these "asymptotic" excitations.

Comparing the answer with (11), (9), in this approximation we obtain

$$Z_\phi = 1, \qquad Z_\alpha = \left(\frac{R}{r_0}\right)^{\delta}. \tag{13}$$

In the first order in the perturbation theory, the renormalization group method will give us nothing new, and I give it just for illustration. We expand Z_α in the small parameter δ:

$$Z_\alpha = 1 + \delta\log\frac{R}{r_0},$$

We will assume the cutoff parameter r_0 and the seed constants α_0, δ_0 to be constant, and the logarithm of the infrared scale R to be renormalization group "time". Then

$$\frac{d\alpha}{dt} = -\alpha_0\delta, \qquad t = \log R.$$

In the first order in δ, we must assume $\alpha_0 = \alpha$. Therefore, we obtain the equation

$$\frac{d\alpha}{dt} = -\alpha\delta,$$

which has a solution

$$\alpha \sim R^{-\delta},$$

in consistency with (13). This is natural, since the result (13) is exact, when the coupling constant μ is much smaller than the cutoff scale $r_0^{\beta^2-2}$, that is, $\alpha \ll 1$. The exactness of this answer is due to the use of the conformal perturbation theory.

Formally, renormalization-group trajectories look like on Fig. 1. The trajectories are drawn in the plane

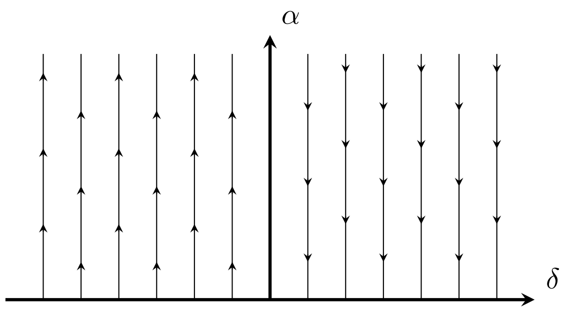

Figure 1: Renormalization-group trajectories in the first order of the perturbation theory. The arrows point in the direction of increasing scale R.

(δ, α). We see that for $\delta < 0$, the asymptotic freedom takes place: the coupling parameter α tends to zero with decreasing scale and grows unlimitedly with increasing scale. In the same way, the diagram looks in the plane of the seed parameters (δ_0, α_0), but in this case with respect to the ultraviolet "time" $\log r_0$.

Now consider the higher orders in the perturbation theory. In the second order of the perturbation theory we have

$$\begin{aligned}\frac{1}{2}\langle\phi(x)\phi(x')S_1^2[\phi]\rangle_{0,\mathrm{c}} &= \frac{\alpha_0^2 r_0^{2\delta_0}}{2}\int d^2y_1\, d^2y_2\, \langle\phi(x)\phi(x')\, {:}\cos\beta_0\phi(y_1){:}\ {:}\cos\beta_0\phi(y_2){:}\,\rangle_{0,\mathrm{c}} \\ &= \alpha_0^2\beta_0^2 r_0^{2\delta_0}\int d^2y_1\, d^2y_2\, \Big(\langle\phi(x)\phi(y_1)\rangle_0\langle\phi(x')\phi(y_2)\rangle_0\langle\, {:}\sin\beta_0\phi(y_1){:}\ {:}\sin\beta_0\phi(y_2){:}\,\rangle_0 \\ &\qquad - \langle\phi(x)\phi(y_1)\rangle_0\langle\phi(x')\phi(y_1)\rangle_0\Big(\langle\, {:}\cos\beta_0\phi(y_1){:}\ {:}\cos\beta_0\phi(y_2){:}\,\rangle_0 - R_0^{-2\beta_0^2}\Big)\Big).\end{aligned}$$

The factor $R_0^{-2\beta_0^2}$ in the last term is related to the fact that each vertex contributes to the diagram $R_0^{-\beta_0^2} = \langle\, {:}e^{\pm i\beta\phi}{:}\,\rangle_0$. From this we immediately find

$$\begin{aligned}-\frac{1}{4\pi}\Sigma^{(2)}(x) &= \alpha_0^2\beta_0^2 r_0^{2\delta_0}\Big(\langle\, {:}\sin\beta_0\phi(x){:}\ {:}\sin\beta_0\phi(0){:}\,\rangle_0 - \beta_0^2 R_0^{-2\beta_0^2}\langle\phi(x)\phi(0)\rangle_0 \\ &\qquad - \delta(x)\int d^2y\,\Big(\langle\, {:}\cos\beta_0\phi(0){:}\ {:}\cos\beta_0\phi(y){:}\,\rangle_0 - R_0^{-2\beta_0^2}\Big)\Big) \\ &= \frac{\alpha_0^2\beta_0^2 r_0^{2\delta_0}}{2R_0^{2\beta_0^2}}\Bigg(\left(\frac{R_0}{x}\right)^{2\beta_0^2} - \left(\frac{x}{R_0}\right)^{2\beta_0^2} - 2\beta_0^2\log\frac{R_0^2}{x^2} \\ &\qquad - \delta(x)\int d^2y\,\left(\left(\frac{R_0}{y}\right)^{2\beta_0^2} + \left(\frac{y}{R_0}\right)^{2\beta_0^2} - 2\right)\Bigg).\end{aligned}$$

The term with $\langle\phi(x)\phi(0)\rangle_0$ subtracts the one-particle-reducible contribution. In the momentum representation, we have

$$\begin{aligned}\Sigma^{(2)}(p^2) = -2\pi\alpha_0^2\beta_0^2 r_0^{2\delta_0}\Big(&\int d^2x\,(e^{\mathrm{i}px}-1)x^{-2\beta_0^2} \\ &- R_0^{-4\beta_0^2}\int d^2x\,(e^{\mathrm{i}px}+1)x^{2\beta_0^2} - 2\beta_0^2 R_0^{-2\beta_0^2}G_0(p^2) + 2R_0^{2-2\beta_0^2}\Big).\end{aligned} \quad (14)$$

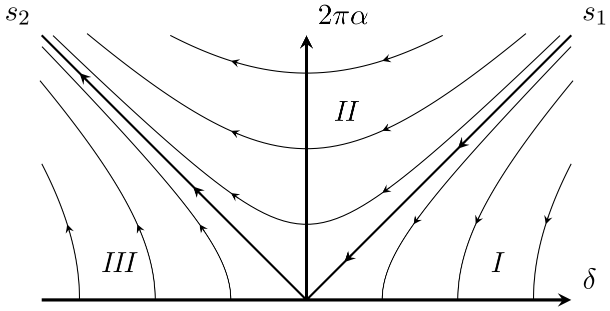

Figure 2: Renormalization-group trajectories in the second order of the perturbation theory.

The second line tends to zero as $R_0 \to \infty$. The first integral contributes only to Σ_1:

$$\Sigma^{(2)}(p^2) = \pi\alpha_0^2\beta_0^2 r_0^{2\delta_0} \int d^2x\,(px)^2 x^{-2\beta_0^2} + O(p^4) \simeq \pi^2\alpha_0^2\beta_0^2 p^2 \log\frac{R_0}{r_0} + O(p^4) \quad (\delta_0 \ll 1). \tag{15}$$

Under the logarithm sign, we can replace R_0 with R. From this in the second order we obtain

$$Z_\phi = 1 - \pi^2\alpha_0^2\beta_0^2 \log\frac{R}{r_0}, \qquad Z_\alpha = 1 + \delta_0 \log\frac{R}{r_0}. \tag{16}$$

The renormalization group equations in the second order of α and in the leading order of δ have the form:

$$\frac{d\alpha}{dt} = -\delta\alpha, \qquad \frac{d\delta}{dt} = -4\pi^2\alpha^2, \qquad t = \log R. \tag{17}$$

Note that these equations can be written as

$$\frac{d(2\pi\alpha \mp \delta)}{dt} = \pm 2\pi\alpha(2\pi\alpha \mp \delta). \tag{17a}$$

This means that the straight lines $2\pi\alpha = \pm\delta$ are renormalization-group trajectories. These lines divide the half-plane (δ, α) into three parts. In each of these parts it is not difficult to qualitatively construct renormalization-group trajectories (Fig. 2).

What is this diagram talking about? First of all, region I represents the region of the low-temperature ("molecular") phase. The points on the right semiaxis δ are stable fixed points. This means that in the infrared limit the coupling constant α tends to zero and the theory is described by a free massless boson. The separatrix s_1, separating regions I and II, is a line of critical points. Any point on the separatrix flows to the point $\alpha = \delta = 0$. Regions II and III are regions of the high-temperature ("plasma") phase, and region III is asymptotically free, that is, it is described by the theory of the free field in the ultraviolet limit. With increasing scale, the interaction constant α grows, so that the excitations in the infrared region are in no way connected with the initial free boson. In fact, these excitations are massive particles, and massive particles are also present at the point $\beta^2 = 2$. This follows from the fact that, if we turn on even a weak interaction at this point, we immediately get into region II, in which the trajectories go to infinity in the infrared limit. On the other hand, at this point the theory is not described by the massless theory on small scales either: the trajectories in region II also begin at infinity.

How do the trajectories behave further? It cannot be obtained from the perturbation theory, because the perturbation theory makes it possible to obtain a renormalization group only in the case of an almost dimensionless coupling constant ($\delta \ll 1$). It is conjectured that in the region $1 < \beta^2 < 2$ ($-1 < \delta < 0$) the trajectories tend to the line $\beta^2 = 1$, which corresponds, as we have seen, to the free-fermion point.

Bibliography

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Problems

1. Show that the second line in (14) is negligible and derive (15).

2. Solve the system of equations (17) exactly. Find the algebraic equation for the trajectories and their explicit dependence on $t = \log R$.

3. Let $\varphi(z)$ be the chiral (right) field. Let us introduce the currents of scale dimension 1:

$$J^0(z) = \frac{\mathrm{i}}{\sqrt{2}}\partial\varphi(z), \qquad J^\pm(z) = e^{\pm \mathrm{i}\sqrt{2}\varphi(z)}.$$

Show that these currents have operator product expansions (these operator product expansions are equivalent to the relations of the Kac–Moody $\widehat{sl}_2$ algebra with the central charge equal to 1):

$$\begin{aligned} J^0(z')J^0(z) &= \frac{1/2}{(z'-z)^2} + O(1), \\ J^+(z')J^-(z) &= \frac{1}{(z'-z)^2} + \frac{2J^0(z)}{z'-z} + O(1), \\ J^\pm(z')J^0(z) &= \mp\frac{J^\pm(z)}{z'-z} + O(1). \end{aligned}$$

All other operator product expansions are regular.

4. Let J^α be the currents from the previous problem. Find the operator product expansions of these currents with the operators $V_\pm(z) = e^{\pm\frac{\mathrm{i}}{\sqrt{2}}\varphi(z)}$.

5*. In the theory of a free boson field, show that the energy-momentum tensor has only two nonzero components:

$$T_{zz} = -\frac{T(z)}{2\pi} = \frac{(\partial\phi)^2}{4\pi}, \qquad T_{\bar z\bar z} = -\frac{\bar T(z)}{2\pi} = \frac{(\bar\partial\phi)^2}{4\pi}.$$

In the quantum case, the product must be replaced with the normal product $:\ldots:$.

Show that the components of the energy-momentum tensor have the following operator product expansion:

$$T(z')T(z) = \frac{1/2}{(z'-z)^4} + \frac{2T(z)}{(z'-z)^2} + \frac{\partial T(z)}{z'-z} + O(1).$$

Show that

$$T(z')V_\alpha(z) = \frac{\Delta_\alpha V_\alpha(z)}{(z'-z)^2} + \frac{\partial V_\alpha(z)}{z'-z} + O(1), \qquad V_\alpha(z) = e^{i\alpha\varphi(z)},$$

with $\Delta_\alpha = \alpha^2/2$ being the scaling dimension of the operator $V_\alpha(z)$.