Lecture 1
$O(2)$-model and Berezinskii–Kosterlitz–Thouless transition

In these lectures we will often consider the models in two-dimensional space-time with the action

$$S[\boldsymbol{n}] = \frac{1}{2g}\int d^2x\,(\partial_\mu \boldsymbol{n})^2, \qquad \boldsymbol{n}^2 \equiv \sum_{i=1}^{N} n_i^2 = 1, \tag{1}$$

which are called $\boldsymbol{n}$-field models or $O(N)$-models. These models have an explicit $O(N)$ symmetry, which is the symmetry of rotations of a sphere. They belong to a wide class of sigma-models, that is, models in which the fields lie on manifolds.

Now we will consider the simplest model in this series, the $O(2)$-model. It elementarily linearizes. Let

$$n_1 = \cos\varphi, \qquad n_2 = \sin\varphi.$$

Then

$$S[\varphi] = \frac{1}{2g}\int d^2x\,(\partial_\mu\varphi)^2, \tag{2}$$

$$\varphi(x) \sim \varphi(x) + 2\pi. \tag{3}$$

The last line means that we consider the values φ and $\varphi + 2\pi$ of the field equivalent.

One would think that we have a massless field with correlation functions of for operators consistent with (3) decaying power-like:

$$\langle e^{im\varphi(x')} e^{in\varphi(x)}\rangle \sim \left(-(x'-x)^2\right)^{\frac{g}{4\pi}mn}, \qquad m, n \in \mathbb{Z}. \tag{4}$$

In fact, this is not at all the case, and the result substantially depends on the value of the constant g. To make sure of it let us consider the classical solution of the field equations

$$\nabla^2\varphi = 0 \tag{5}$$

in the Euclidean space. It admits solutions of the form

$$\varphi_{\vec q\vec x}(x) = \sum_{a=1}^{n} q_a \operatorname{Im}\log(z - z_a) = \sum_{a=1}^{n} \frac{q_a}{2i}\log\frac{z - z_a}{\bar z - \bar z_a}, \qquad q_a \in \mathbb{Z}, \tag{6}$$

where

$$z = x^1 + ix^2 = x^1 - x^0,$$
$$\bar z = x^1 - ix^2 = x^1 + x^0.$$

Though these solutions have singularities (indefinite values) at the points $z = z_a$, they are very important. They are solutions with n vortices at the points z_a with vorticities q_a. In the simplest case $n = 1$ in the radial coordinates $z - z_1 = re^{i\theta}$ the solution has the form

$$\varphi_{q_1x_1}(x) = q_1\theta.$$

It is important to note that solutions (6) satisfy the equation (5) even at the points $x = x_a$. Indeed,

$$\partial_\mu\partial^\mu \frac{1}{2i}\log\frac{z}{\bar z} = \partial_\mu\partial^\mu \operatorname{arctg}\frac{x^2}{x^1} = -\epsilon_{\mu\nu}\partial_\mu\frac{x^\nu}{r^2} = \epsilon^{\mu\nu}\partial_\mu\partial_\nu\log\frac{1}{r}.$$

Then for any smooth, bounded and decreasing fast enough function $\varphi(x)$ we have

$$\int d^2x\,\varphi(x)\partial_\mu\partial^\mu\frac{1}{2i}\log\frac{z}{\bar z} = \int d^2x\,(\epsilon^{\mu\nu}\partial_\mu\partial_\nu\varphi(x))\log\frac{1}{r} = 0,$$

since the integral of $\log r$ converges at $x = 0$. Hence we immediately obtain

$$\int d^2x\,\partial^\mu\varphi\,\partial_\mu\varphi_{\vec q\vec x} = 0. \tag{7}$$

Let us calculate the value of the action on the vortex solutions (6):

$$S[\varphi_{\vec{q}\vec{x}}] = \frac{2}{g}\int d^2x\, \partial\varphi_{\vec{q}\vec{x}}\,\bar\partial\varphi_{\vec{q}\vec{x}} = \frac{1}{2g}\int d^2x \sum_{a,b} \frac{q_a q_b}{(z-z_a)(\bar z-\bar z_b)}$$

$$= \frac{1}{2g}\left(\sum_a q_a^2 \int \frac{d^2x}{|z-z_a|^2} + \sum_{a<b} q_a q_b \int d^2x\, \frac{(z-z_a)(\bar z-\bar z_b)+(\bar z-\bar z_a)(z-z_b)}{|z-z_a|^2|z-z_b|^2}\right).$$

The first integral is taken easily, but diverges on both large and small scales:

$$\int \frac{d^2x}{|z-z_a|^2} \simeq 2\pi \int_{r_0}^{R} \frac{dr}{r} = 2\pi \log\frac{R}{r_0},$$

where R and r_0 are infrared and ultraviolet cutoff parameters respectively. The second integral only diverges on large scales. We have

$$\int d^2x\, \frac{(z-z_a)(\bar z-\bar z_b)+(\bar z-\bar z_a)(z-z_b)}{|z-z_a|^2|z-z_b|^2} = 2\pi\log\frac{R^2}{|z_a-z_b|^2}.$$

Substituting these formulas into the integral for the action, we obtain

$$S[\varphi_{\vec{q}\vec{x}}] = \frac{1}{2g}\left(\pi\sum_a q_a^2 \log\frac{R^2}{r_0^2} + 2\pi\sum_{a<b} q_a q_b \log\frac{R^2}{|z_a-z_b|^2}\right) \tag{8}$$

$$= \frac{\pi}{2g}\left(\sum_a q_a\right)^2 \log R^2 - \frac{\pi}{2g}\sum_a q_a^2 \log r_0^2 + \frac{1}{2g}\sum_{a<b} q_a q_b\, 2\pi\log\frac{1}{|z_a-z_b|^2}. \tag{9}$$

The first term tends to infinity when the size of the system grows, if the expression in parentheses is nonzero. This means that in a large system the neutrality condition must be satisfied:

$$\sum_a q_a = 0. \tag{10}$$

The second term in (9) has an ultraviolet divergence. If we regularize the theory somehow in a natural way, for example, consider it as a limit of the theory with the action

$$S[\phi] = \int d^2x\, \left(|\partial_\mu\phi|^2 - \frac{\lambda}{4}(|\phi|^2-\phi_0^2)^2\right),$$

this term will be finite and will depend on the structure of the vortex core. Below we see that the value r_0 does not significantly affect the result.

Let us now try to write a (Euclidean) functional integral in the form

$$Z[J] = \sum_{n=0}^{\infty} \frac{r_0^{-2n}}{n!} \sum_{\substack{q_1,\dots,q_n\\ q_1+\dots+q_n=0}} \int d^2x_1\cdots d^2x_n \int D\varphi\, e^{-S[\varphi+\varphi_{\vec{q}\vec{x}}]-(J,\varphi+\varphi_{\vec{q}\vec{x}})}, \tag{11}$$

where the integral is now taken over regular fields φ without any identification. The $1/n!$ factor comes from the fact that the solution (9) does not change under permutations $z_a \leftrightarrow z_b$, $q_a \leftrightarrow q_b$. Thus, the summation $\sum_{q_1,\dots,q_n}\int d^2x_1\cdots d^2x_n$ takes into account the same configuration $n!$ times. The factor r_0^{-2n} is added in order to make the integral dimensionless. One can imagine that vortices can occupy not any positions, but are located in cells of the size $\sim r_0$.

We calculate the action against the background of a multivortex solution:

$$S[\varphi+\varphi_{\vec{q}\vec{x}}] = S[\varphi_{\vec{q}\vec{x}}] + S[\varphi] + \frac{1}{g}\int d^2x\, \partial^\mu\varphi\, \partial_\mu\varphi_{\vec{q}\vec{x}}.$$

The first term is given by (9). The integral in the last term vanishes due to (7). Therefore,

$$Z[J] = Z_0[J] \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\substack{q_1,\dots,q_n \\ q_1+\dots+q_n=0}} r_0^{\frac{\pi}{g}\sum q_a^2 - 2n} \int d^2x_1 \cdots d^2x_n \prod_{a<b} |z_a - z_b|^{2\frac{\pi}{g} q_a q_b} \, e^{-(J,\varphi_{\vec{q}\vec{x}})}, \tag{12}$$

$$Z_0[J] = \int D\varphi \, e^{-S[\varphi]-(J,\varphi)}. \tag{13}$$

From the identification (3) it follows that we can only consider sources J of the kind

$$J_{\vec{J}\vec{y}}(x) = -i \sum_{j=1}^{k} J_j \delta(x - y_j), \qquad J_i \in \mathbb{Z}. \tag{14}$$

Then

$$\begin{aligned} Z[J_{\vec{J}\vec{y}}] = & \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\substack{q_1,\dots,q_n \\ q_1+\dots+q_n=0}} r_0^{\frac{\pi}{g}\sum_a q_a^2 + \frac{g}{4\pi}\sum_j J_j^2 - 2n} \int d^2x_1 \cdots d^2x_n \prod_{a<b} |z_a - z_b|^{\frac{2\pi}{g} q_a q_b} \\ & \times \prod_{a,j} \left(\frac{w_j - z_a}{\bar{w}_j - \bar{z}_a} \right)^{q_a J_j/2} \prod_{j<j'} |w_j - w_{j'}|^{\frac{g}{2\pi} J_j J_{j'}}, \end{aligned} \tag{15}$$

where $w_j = y_j^1 + i y_j^2$.

We got something like a partition function of a plasma with arbitrary particle charges (and the energy of a charged state proportional to the squared charge). Source of the φ particles is related to plasma particles in a complicated way, but, in principle, it can be expected that for small coupling constants g ("low temperatures") plasma recombines and correlation functions remain power-like, while for large g ("high temperature") there is a Debye screening and correlation functions decrease exponentially, which means that the theory is massive. Such transition in a coupling constant is called the *Berezinskii–Kosterlitz–Thouless (BKT) transition.*

Surely, we cannot summarize the entire perturbation theory series. However, the BKT transition point can be exactly determined. Indeed, plasma does not form when vortices are held in a finite volume, that is, all the integrals, except one (over the "center of mass"), are infrared convergent at large values of n. Moreover, due to the neutrality condition

$$\sum_{a<b} q_a q_b = \frac{1}{2} \sum_{a \neq b} q_a q_b = \frac{1}{2} \left(\sum_a q_a \right)^2 - \frac{1}{2} \sum_a q_a^2 = -\frac{1}{2} \sum_a q_a^2 \leq -\frac{n}{2}.$$

Hence we find that all integrals converge for

$$2\frac{\pi}{g}\left(-\frac{n}{2}\right) + 2(n-1) < 0.$$

At large n we find that the massless phase corresponds $g < g_{\mathrm{BKT}}$ with

$$g_{\mathrm{BKT}} = \frac{\pi}{2}. \tag{16}$$

At $g > g_{\mathrm{BKT}}$ vortices do not hold and the system behaves like a plasma with a finite correlation length. It should be noted that the answer is independent from the ultraviolet cutoff parameter r_0, so that the phase transition takes place for an arbitrarily small vortex core. Note that the condition (16) is just the condition under which the dimensional factor $r_0^{\frac{\pi}{g}\sum q_a^2 - 2n}$ disappears for a system of simple ($q = \pm 1$) vortices. Since the theory has no dimensional parameters except r_0, the correlation length is proportional to r_0, and thus, even the "ideal" $O(2)$-models has no chance to avoid the phase transition. Qualitatively, this can be explained by the fact that the small statistical weight of vortex states is more than overcome by the large volume of the phase space.

Expression (15) can be rewritten differently by introducing a new field $\phi(x)$. Notice that

$$\nabla^2 \frac{1}{4\pi} \log|x|^2 = \delta(x). \tag{17}$$

and thus $\log \frac{R^2}{|x|^2}$ is the propagator of a free massless boson field:

$$S_0[\phi] = \frac{1}{8\pi} \int d^2x \, (\partial_\mu \phi)^2. \tag{18}$$

Since the equations of motion in this model have the form

$$\partial_\mu \partial^\mu \phi = 0,$$

we can introduce a dual field $\tilde{\phi}$ by the condition

$$\partial_\mu \tilde{\phi} = \epsilon_{\mu\nu} \, \partial^\nu \phi, \qquad \epsilon_{01} = -\epsilon_{10} = 1, \tag{19m}$$

in the Minkowski space, or

$$\partial_\mu \tilde{\phi} = -i\epsilon_{\mu\nu} \, \partial^\nu \phi, \qquad \epsilon_{12} = -\epsilon_{21} = 1, \tag{19e}$$

in the Euclidean space, or

$$\partial\tilde{\phi} = \partial\phi, \qquad \bar{\partial}\tilde{\phi} = -\bar{\partial}\phi. \tag{20}$$

Though these formulas have literal meaning only on solutions of the equations of motion, it is easy to show that even on correlation functions these equalities are not without meaning. Indeed, introduce the fields $\phi_R(z)$ and $\phi_L(\bar{z})$ by the equations

$$\begin{aligned} \phi(x) &= \phi_R(z) + \phi_L(\bar{z}), \\ \tilde{\phi}(x) &= \phi_R(z) - \phi_L(\bar{z}). \end{aligned} \tag{21}$$

Then the correlation functions

$$\langle \phi_R(z)\phi_R(z')\rangle_0 = \log \frac{R}{z - z'}, \qquad \langle \phi_L(\bar{z})\phi_L(\bar{z}')\rangle_0 = \log \frac{R}{\bar{z} - \bar{z}'}, \qquad \langle \phi_R(z)\phi_L(\bar{z}')\rangle_0 = 0 \tag{22}$$

are consistent with the theory.

Next, we need the correlation functions of the exponential operators. Since these correlation functions contain infinite factors, we simply exclude them by defining the renormalized exponential operators:

$$e^{i\alpha\phi_{R,L}} = r_0^{\alpha^2/2} :e^{i\alpha\phi_{R,L}}:, \qquad e^{i\alpha\phi} = r_0^{\alpha^2} :e^{i\alpha\phi}:, \qquad e^{i\alpha\tilde{\phi}} = r_0^{\alpha^2} :e^{i\alpha\tilde{\phi}}:. \tag{23}$$

Then the exponential operators $:e^{(\cdots)}:$ are no more dimensionless, and acquire the dimensions $d = \alpha^2/2$ for chiral operators and α^2 for exponents of the fields $\phi, \tilde{\phi}$. These dimensions coincide with the *scaling dimensions* of the operators. By definition, the we have a system of operators O_i with scaling dimensions d_i, if all their correlation functions are invariant with respect to the substitutions

$$O_i(x) \to s^{d_i} O_i(sx)$$

in all operators simultaneously.

The correlation functions of the operator exponents in such a model are equal to

$$\begin{aligned} \langle :e^{i\alpha_1\phi_R(z_1)}: \cdots :e^{i\alpha_n\phi_R(z_n)}:\rangle_0 &= R^{-\frac{1}{2}\left(\sum_a \alpha_a\right)^2} \prod_{a<b} (z_a - z_b)^{\alpha_a\alpha_b}, \\ \langle :e^{i\alpha_1\phi_L(\bar{z}_1)}: \cdots :e^{i\alpha_n\phi_L(\bar{z}_n)}:\rangle_0 &= R^{-\frac{1}{2}\left(\sum_a \alpha_a\right)^2} \prod_{a<b} (\bar{z}_a - \bar{z}_b)^{\alpha_a\alpha_b}. \end{aligned} \tag{24}$$

A more accurate description of this renormalization procedure is given at the end of the lecture in the Explanation. In the limit $R \to \infty$ the r.h.s. are only nonzero if

$$\sum_{a=1}^n \alpha_a = 0. \tag{25}$$

Then we have

$$\left\langle \prod_{j=1}^{k} e^{i\beta_j\tilde\phi(y_j)} \prod_{a=1}^{n} e^{i\alpha_a\phi(x_a)} \right\rangle_0$$
$$= r_0^{\sum_a \alpha_a^2+\sum_j \beta_j^2} \prod_{a<b} |z_a-z_b|^{2\alpha_a\alpha_b} \prod_{j<j'} |w_j-w_{j'}|^{2\beta_j\beta_{j'}} \prod_{a,j} \left(\frac{w_j-z_a}{\bar w_j-\bar z_a}\right)^{\alpha_a\beta_j} \times \begin{cases} 1, & \sum\alpha_a=\sum\beta_j=0; \\ 0 & \text{otherwise.}\end{cases} \tag{26}$$

This expression exactly matches the integrand in (15) for

$$\alpha_a = \sqrt{\frac{\pi}{g}}\, q_a, \qquad \beta_j = \sqrt{\frac{g}{4\pi}}\, J_j. \tag{27}$$

From this we obtain

$$Z[J_{\vec J\vec y}] = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\substack{q_1,\dots,q_n\\ q_1+\dots+q_n=0}} r_0^{-2n} \int d^2x_1\cdots d^2x_n \left\langle \prod_{j=1}^{k} e^{i\sqrt{\frac{g}{4\pi}}\,J_j\tilde\phi(y_j)} \prod_{a=1}^{n} e^{i\sqrt{\frac{\pi}{g}}\,q_a\phi(x_a)} \right\rangle_0. \tag{28}$$

Note that the expression under the sign of the integral is remarkably symmetric with respect to the replacements

$$g\leftrightarrow(2\pi)^2g^{-1}, \qquad k\leftrightarrow n, \qquad q_a\leftrightarrow J_j, \qquad \phi(x)\leftrightarrow\tilde\phi(x).$$

Moreover, the Lagrangian of the free field is written identically in terms of both the fields ϕ and $\tilde\phi$. Thus we can identify

$$\varphi(x) = \sqrt{\frac{g}{4\pi}}\,\tilde\phi(x). \tag{29}$$

Let us make an important approximation, which does not change the properties of the phase transition. Let us neglect the vortices with $|q|>1$, as their contribution decreases with decreasing r_0 faster than the contribution of $|q|$ charge 1 vortices. Then the generating functional can be rewritten as

$$\begin{aligned} Z[J_{\vec J\vec y}] &= \sum_{n=0}^{\infty} \frac{r_0^{-4n}}{(2n)!} \int d^2x_1\cdots d^2x_{2n} \sum_{q_1,\dots,q_{2n}=\pm1} \left\langle \prod_{j=1}^{k} e^{i\sqrt{\frac{g}{4\pi}}\,J_j\tilde\phi(y_j)} \prod_{a=1}^{2n} e^{iq_a\sqrt{\frac{\pi}{g}}\phi(x_a)} \right\rangle_0 \\ &= \sum_{n=0}^{\infty} \frac{r_0^{-4n}}{(2n)!} \int d^2x_1\cdots d^2x_{2n} \left\langle \prod_{j=1}^{k} e^{i\sqrt{\frac{g}{4\pi}}\,J_j\tilde\phi(y_j)} \prod_{a=1}^{2n} \left(e^{i\sqrt{\frac{\pi}{g}}\phi(x_a)} + e^{-i\sqrt{\frac{\pi}{g}}\phi(x_a)}\right) \right\rangle_0 \\ &= \left\langle \prod_{j=1}^{k} e^{i\sqrt{\frac{g}{4\pi}}\,J_j\tilde\phi(y_j)} \exp\left(2r_0^{-2}\int d^2x\,\cos\sqrt{\frac{\pi}{g}}\phi(x)\right) \right\rangle_0 \\ &= \int D\phi\, e^{-S_{\rm SG}[\phi]} \prod_{j=1}^{k} e^{i\sqrt{\frac{g}{4\pi}}\,J_j\tilde\phi(y_j)}, \end{aligned} \tag{30}$$

where

$$S_{\rm SG}[\phi] = \int d^2x \left(\frac{(\partial_\mu\phi)^2}{8\pi} - \mu\,{:}\cos\beta\phi{:}\right) \tag{31}$$

is the action of the sine-Gordon model with the parameters

$$\beta = \sqrt{\frac{\pi}{g}}, \qquad \mu = 2r_0^{\frac{\pi}{g}-2}. \tag{32}$$

We have written the action in terms of the renormalized exponents. In what follows we will usually omits the symbols $:\cdots:$ and mean by exponents just the renormalized ones by default.

We will study the sine-Gordon model in more detail next time, but for now we introduce several important concepts. We will consider the sine-Gordon model as perturbation of the free massless boson. Then the scaling dimension of the perturbation operator will be equal

$$d_{\rm pert} = \beta^2.$$

When $d_{\text{pert}} < 2$ the perturbation is called *relevant*. It significantly changes the behavior of the system at large scales and does not change at small ones. When $d_{\text{pert}} > 2$ perturbation is called *irrelevant* and does not change qualitatively the infrared behavior. The case $d_{\text{pert}} = 2$ is called *marginal*. In the case of the sine-Gordon model, just this case corresponds to the BKT transition point.

Explanation

This explanation concerns the definition of the exponential operators in the theory of a free scalar field. For simplicity, we restrict ourselves to the functionals of the field $\varphi(z) \equiv \varphi_R(z)$. An expansion can be written for this field

$$\varphi(z) = Q - iP\log z + \sum_{k\neq 0} \frac{a_k}{ik} z^{-k}, \tag{33}$$

where Hermitian operators P, Q and operators of creation-annihilation $a_k = a^+_{-k}$ satisfy the relations

$$[P, Q] = -i, \qquad [a_k, a_l] = k\delta_{k+l,0}. \tag{34}$$

If we determine the vacuum $|0\rangle$ by the conditions

$$P|0\rangle = a_k|0\rangle = 0 \quad (k > 0), \tag{35}$$

it is easy to check that

$$\langle\varphi(z')\varphi(z)\rangle = \langle Q^2\rangle + \langle\varphi(z')\varphi(z)\rangle_* = \langle Q^2\rangle + \log\frac{1}{z'-z}, \tag{36}$$

The indefinite expression $\langle Q^2\rangle$ can be identified with an infrared term $\log R$ in (22). It is also easy to see that the standard normal ordering that puts P to the right of Q and a_k $(k > 0)$ to the right of a_{-k} meets the condition

$$\varphi(z_1)\varphi(z_2) = {:}\varphi(z_1)\varphi(z_2){:} + \langle\varphi(z_1)\varphi(z_2)\rangle_*.$$

More generally, the normal ordering can be specified by the recursion relation

$$:\varphi(z_1)\cdots\varphi(z_n):\varphi(z) = :\varphi(z_1)\cdots\varphi(z_n)\varphi(z): + \sum_{i=1}^{n} :\varphi(z_1)\overset{\widehat{i}}{\cdots}\varphi(z_n):\langle\varphi(z_i)\varphi(z)\rangle_* \tag{37}$$

with the initial condition

$$:1: = 1. \tag{38}$$

Here is the index $\hat{i}$ above the ellipsis means that the ith factor is excluded from the normal product. From this definition it is easy to derive the identity

$$:\varphi(z_1)\cdots\varphi(z_m): :\varphi(w_1)\cdots\varphi(w_n): = \\ = \sum_{k=0}^{\min(m,n)} \sum_{\substack{1\leq i_1<\cdots<i_k\leq m\\ 1\leq j_1,\ldots,j_k\leq n}} :\varphi(z_1)\overset{\widehat{i_1}\ldots\widehat{i_k}}{\cdots}\varphi(z_m)\varphi(w_1)\overset{\widehat{j_1}\ldots\widehat{j_k}}{\cdots}\varphi(w_n): \prod_{l=1}^{k}\langle\varphi(z_{i_l})\varphi(w_{j_l})\rangle_*. \tag{39}$$

Return to the operator exponents $e^{i\alpha\varphi(z)}$. Operator products of formal exponents

$$e^{i\alpha_1\varphi(z_1)}e^{i\alpha_2\varphi(z_2)} = e^{-\frac{1}{2}\alpha_1\alpha_2[\varphi(z_1),\varphi(z_2)]}e^{i\alpha_1\varphi(z_1)+i\alpha_2\varphi(z_2)} \tag{40}$$

rather poorly defined, because they contain a poorly defined commutator. Correlation functions of formal exponents

$$\langle e^{i\alpha_1\varphi(z_1)}\cdots e^{i\alpha_N\varphi(z_N)}\rangle = \left(\frac{r_0}{R}\right)^{\frac{1}{2}\sum_i\alpha_i^2}\prod_{i<j}\left(\frac{z_i-z_j}{R}\right)^{\alpha_i\alpha_j} \tag{41}$$

contain ultraviolet divergences. Thus, formal exponents are poorly defined.

Normal exponents are well defined, all of their correlators are ultraviolet finite. It can be verified that

$$\langle :e^{i\alpha_1\varphi(z_1)+\cdots+i\alpha_n\varphi(z_n)}: \rangle = \langle e^{i(\alpha_1+\cdots+\alpha_N)Q}\rangle = R^{-\frac{1}{2}(\sum\alpha_i)^2}. \tag{42}$$

The operator products of normal exponents have the form

$$:e^{i\alpha_1\varphi(z_1)}: \, :e^{i\alpha_2\varphi(z_2)}: \, = (z_1-z_2)^{\alpha_1\alpha_2} :e^{i\alpha_1\varphi(z_1)+i\alpha_2\varphi(z_2)}: . \tag{43}$$

From this it is not difficult to find the correlation functions

$$\langle :e^{i\alpha_1\varphi(z_1)}: \cdots :e^{i\alpha_N\varphi(z_N)}: \rangle = R^{-\frac{1}{2}(\sum\alpha_i)^2} \prod_{i<j}(z_i-z_j)^{\alpha_i\alpha_j}. \tag{44}$$

Comparing this to (41), we see that

$$e^{i\alpha\varphi(z)} = r_0^{\alpha^2/2} :e^{i\alpha\varphi(z)}: ,$$

that is, normal exponents depending on one field $\varphi(z)$, are nothing but the renormalized versions of the full operator exponents defined in (23). Outside of this explanation, in order not to clutter up formulas, we will omit the normal product sign.

The expression (44) explicitly contains infrared cutoff R, but has a good limit as $R\to\infty$:

$$\langle :e^{i\alpha_1\varphi(z_1)}: \cdots :e^{i\alpha_N\varphi(z_N)}: \rangle = \begin{cases} \prod_{i<j}(z_i-z_j)^{\alpha_i\alpha_j}, & \text{if } \sum_i \alpha_i = 0; \\ 0, & \text{if } \sum_i \alpha_i \neq 0. \end{cases} \tag{45}$$

In particular, on the infinite plane

$$\langle :e^{i\alpha_1\varphi(z_1)+\cdots+i\alpha_n\varphi(z_n)}: \rangle = \begin{cases} 1, & \text{if } \sum_i \alpha_i = 0; \\ 0, & \text{if } \sum_i \alpha_i \neq 0. \end{cases} \tag{46}$$

Problems

1. Calculate the space integrals and obtain (8).

2. Derive the formula (17).

3. Show that for the free field ϕ with the action $S_0[\phi]$ the pair correlation function is equal to

$$\langle\phi(x)\phi(y)\rangle = \log\frac{R^2}{(x-y)^2}$$

with some scale of the infrared cutoff R.

4. Find the conditions under which the operators $e^{i\alpha_1\varphi_R(z)+i\beta_1\varphi_L(\bar z)}$ and $e^{i\alpha_2\varphi_R(z')+i\beta_2\varphi_L(\bar z')}$ are mutually local, i.e. possess correlation functions single-valued when x goes around x'.

5*. Suppose the field $\phi(x)$ with the action $S_0[\phi]$ is defined on a circle of radius R ($\phi\sim\phi+2\pi R$) and lives on spatial circle ($x^1\sim x^1+2\pi$) with periodic boundary conditions. Show that the theory is equivalent to the field theory $\tilde\phi(x)$ defined on a circle of radius $2/R$ (*T-duality*). To solve the problem you can use the expansion in modes in the Hamiltonian formalism. In this case, it should be noted that while traversing the spatial cycle the field may change by an integer number periods $2\pi R$ (winding number). The duality transformation interchanges the winding number and the quantum number of momentum.