# Lecture 4
# Solving Bethe equations

A mini-course “Solvable lattice models and Bethe Ansatz”
(Ariel University, spring 2021)

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# Bethe equations

Let

$$s(u)=\begin{cases}\sin u & \text{for } |\Delta|<1;\\ \operatorname{sh} u & \text{for } \Delta<-1;\end{cases}\qquad c(u)=\begin{cases}\cos u & \text{for } |\Delta|<1;\\ \operatorname{ch} u & \text{for } \Delta<-1.\end{cases}$$

The explicit form of the Bethe equations:

$$\left(\frac{s(u_i)}{s(\lambda-u_i)}\right)^N=\prod_{\substack{j=1\\(j\neq i)}}^{n}\frac{s(u_i-u_j+\lambda)}{s(u_i-u_j-\lambda)}. \tag{1}$$

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Let

$$u_i=\frac{\lambda}{2}+\mathrm{i}v_i,\quad e^{\mathrm{i}p(v)}=\frac{s(\lambda/2+\mathrm{i}v)}{s(\lambda/2-\mathrm{i}v)},\quad e^{\mathrm{i}\theta(v)}=\frac{s(\lambda+\mathrm{i}v)}{s(\lambda-\mathrm{i}v)}.$$

The variables $v_i$ are defined in such a way that $|z_i|=1$ for real values of $v_i$.

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$$u_i = \frac{\lambda}{2} + \mathrm{i}v_i, \quad e^{\mathrm{i}p(v)} = \frac{s(\lambda/2 + \mathrm{i}v)}{s(\lambda/2 - \mathrm{i}v)}, \quad e^{\mathrm{i}\theta(v)} = \frac{s(\lambda + \mathrm{i}v)}{s(\lambda - \mathrm{i}v)}.$$

The variables $v_i$ are defined in such a way that $|z_i| = 1$ for real values of $v_i$. Take logarithm of the Bethe equations:

$$Np(v_i) = 2\pi I_i + \sum_{j=1}^{n} \theta(v_i - v_j),$$

where $I_i \in \mathbb{Z} + \frac{1}{2}$ if $n \in 2\mathbb{Z}$ and $I_i \in \mathbb{Z}$ if $n \in 2\mathbb{Z} + 1$.

# Solving the Bethe equations for the ground state

Let us obtain the ground state, i.e. the state with the largest eigenvalue of the transfer matrices in the thermodynamic limit. Note that the XXZ one-particle energy

$$\epsilon(v) = 2\Delta - 2\cos p(v)$$

is an even function, $\epsilon(-v) = \epsilon(v)$ with an absolute minimum at $v = 0$ and monotonous for $0 \le v < \infty$ if $|\Delta| < 1$ and for $0 \le v \le \frac{\pi}{2}$ for $\Delta < -1$. It means that the 'Dirac sea' must be symmetric.

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Taking the thermodynamic limit in a usual way, we obtain the integral equations

$$p'(v) = \rho(v) + \int_{-v_F}^{v_F} \frac{dv'}{2\pi}\,\theta'(v - v')\rho(v'), \qquad \int_{-v_F}^{v_F} \frac{dv}{2\pi}\,\rho(v) = \frac{n}{N}, \tag{2}$$

where $\rho(v) = \frac{2\pi dI}{N dv}$ is the density of particles = density of states.

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We have

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We have

$$p'_k = \frac{\operatorname{sh} \frac{(\pi-\lambda)k}{2}}{\operatorname{sh} \frac{\pi k}{2}}, \qquad \theta'_k = \frac{\operatorname{sh} \frac{(\pi-2\lambda)k}{2}}{\operatorname{sh} \frac{\pi k}{2}} \qquad (|\Delta| < 1);$$

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We have for the density
$$\rho_k = \frac{p'_k}{1 + \theta'_k} = \frac{1}{2 \operatorname{ch} \frac{\lambda k}{2}}$$
in both cases.

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$$\frac{n}{N} = \int_{-\bar{v}_F}^{\bar{v}_F} \frac{dv}{2\pi} \rho(v) = \rho_0 = \frac{1}{2} \quad \Rightarrow \quad \frac{S^z}{N} \to 0.$$

Recall the expression for the eigenvalue

$$\Lambda(u; u_1, \ldots, u_n) = a^N(u) \prod_{i=1}^{n} \frac{a(u_i - u)}{b(u_i - u)} + b^N(u) \prod_{i=1}^{n} \frac{a(u - u_i)}{b(u - u_i)}. \qquad \textbf{(??)}$$

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Now let us calculate the free energy per vertex of the six-vertex model:

$$f=-\lim_{N\to\infty}\frac{\log\Lambda_{\max}(u)}{N}=-\max\Bigg(\log a(u)+\int_{-v_F}^{v_F}\frac{dv}{2\pi}\,\rho(v)\log\frac{a(iv-u+\lambda/2)}{b(iv-u+\lambda/2)},$$
$$\log b(u)+\int_{-v_F}^{v_F}\frac{dv}{2\pi}\,\rho(v)\log\frac{a(u-iv-\lambda/2)}{b(u-iv-\lambda/2)}\Bigg).$$

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$$f = -\lim_{N\to\infty} \frac{\log \Lambda_{\max}(u)}{N} = -\max\Bigg( \log a(u) + \int_{-v_F}^{v_F} \frac{dv}{2\pi}\, \rho(v)(-\mathrm{i})p(\mathrm{i}u + v), \\ \log b(u) + \int_{-v_F}^{v_F} \frac{dv}{2\pi}\, \rho(v)(-\mathrm{i})p(\mathrm{i}(\lambda - u) + v) \Bigg).$$

For $v_F = \bar{v}_F$ we can use the Fourier transform. For $|\Delta| < 1$ we have

$$f = \min\left( -\log a(u) - \int \frac{dk}{k} \rho_{-k} p'_k e^{ku}, -\log b(u) - \int \frac{dk}{k} \rho_k p'_k e^{k(\lambda - u)} \right).$$

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For $v_F=\bar v_F$ we can use the Fourier transform. For $|\Delta|<1$ we have

$$f=\min\left(-\log a(u)-\int\frac{dk}{k}\rho_{-k}p'_k e^{ku},-\log b(u)-\int\frac{dk}{k}\rho_k p'_k e^{k(\lambda-u)}\right).$$

By symmetrizing the we find that the two alternatives coincide, so that

$$\begin{aligned}f&=-\log a(u)-\int_0^\infty\frac{dk}{k}\,\frac{\operatorname{sh}uk\operatorname{sh}\frac{\pi-\lambda}{2}k}{\operatorname{sh}\frac{\pi}{2}k\operatorname{ch}\frac{\lambda}{2}k}\\&=-\log b(u)-\int_0^\infty\frac{dk}{k}\,\frac{\operatorname{sh}(\lambda-u)k\operatorname{sh}\frac{\pi-\lambda}{2}k}{\operatorname{sh}\frac{\pi}{2}k\operatorname{ch}\frac{\lambda}{2}k}.\end{aligned} \qquad (5)$$

# Free energy

In the case $\Delta < -1$ the free energy reads

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Finally, we have

$$\begin{aligned} f &= -\log a(u) - u - \sum_{m=1}^{\infty}\frac{e^{-\lambda m}\,\mathrm{sh}\,2um}{m\,\mathrm{ch}\,\lambda m} \\ &= -\log b(u) - (\lambda - u) - \sum_{m=1}^{\infty}\frac{e^{-\lambda m}\,\mathrm{sh}\,2(\lambda-u)m}{m\,\mathrm{ch}\,\lambda m}. \end{aligned} \tag{6}$$

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What if $v_F < \bar{v}_F$? This case corresponds to general homogeneous six-vertex model with arbitrary $a, a', b, b', c, c'$. The ratio $c/c'$ is inessential, but nonunit rations $a/a'$, $b/b'$ correspond to an external field. They can be related to $v_F$. The integral equations do not have an analytic solution, but can be solved numerically. The two alternatives for the free energy are different.