

# Lecture 3

## Algebraic Bethe Ansatz

A mini-course “Solvable lattice models and Bethe Ansatz”  
(Ariel University, spring 2021)

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Landau Institute for Theoretical Physics,  
Kharkevich Institute for Information Transmission Problems

## $L$ operators and reference state

Recall the definition of the  $L$ -operator:

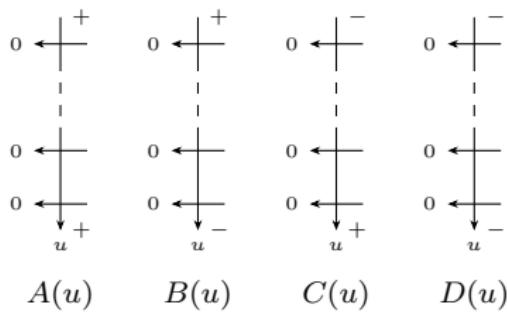
$$L(u) = R_{0N}(u) \dots R_{02}(u)R_{01}(u) = \begin{pmatrix} A(u) & B(u) \\ C(u) & D(u) \end{pmatrix}. \quad (1)$$

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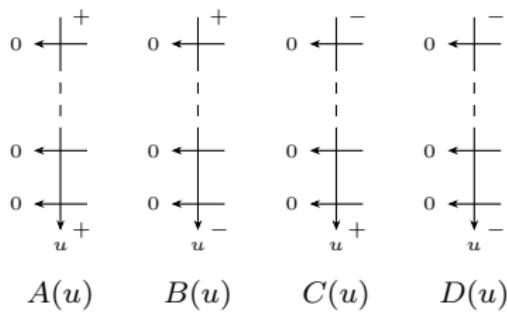


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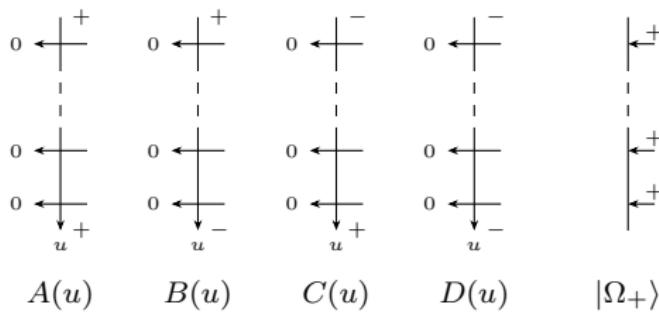
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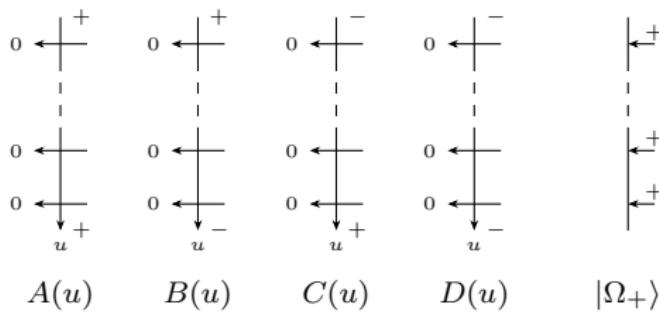
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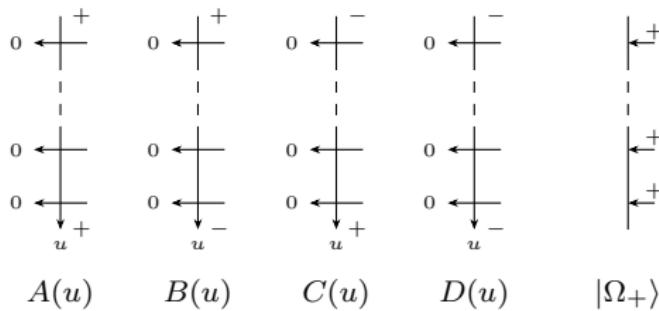
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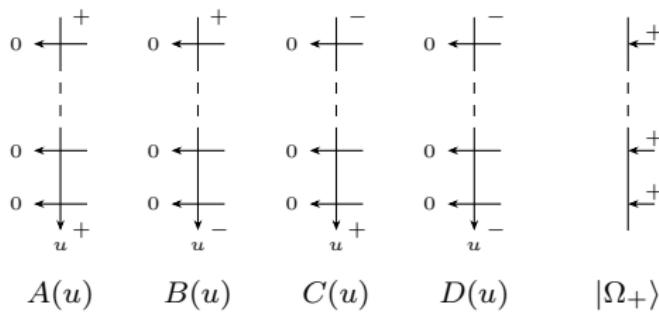
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$$C(u)|\Omega_+\rangle = 0.$$

How to flip spins? By means of  $B(u)$  operators.

Indeed, let

$$|u_1, u_2, \dots, u_k\rangle = B(u_1)B(u_2)\dots B(u_k)|\Omega_+\rangle. \quad (2)$$

Then

$$S^z|u_1, \dots, u_k\rangle = \left(\frac{N}{2} - k\right)|u_1, \dots, u_k\rangle.$$

$n = 1$  case

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Consider the case  $n = 1$ . You see that

$$B(u)|\Omega_+\rangle = \sum_{n=1}^N V_n |c\rangle$$

The diagram illustrates the action of the operator  $B(u)$  on the state  $|\Omega_+\rangle$ . It shows a vertical line with a red double-headed arrow labeled 'a' at the top. Below it is a dashed vertical line with a red double-headed arrow labeled 'c'. To the left of 'c' is a red double-headed arrow labeled  $V_n$ . Below 'c' are two blue double-headed arrows labeled 'b'.

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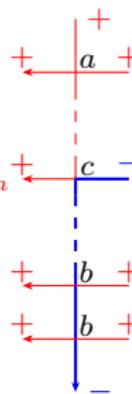
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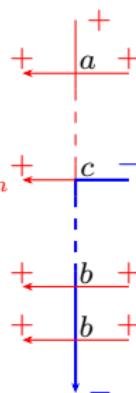
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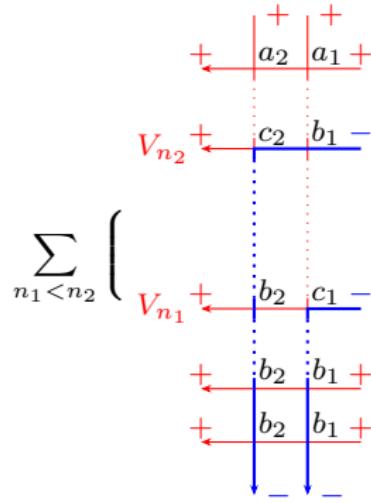
$$B(u)|\Omega_+\rangle \sim \sum_{n=1}^N z^n(u)|n\rangle, \quad z(u) = \frac{b(u)}{a(u)}.$$

It is a Bethe wave function, if  $z^N(u) = 1$ .

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$$\sum_{n_1 < n_2} \left( V_{n_2}^+ \begin{array}{c|c|c} + & a_2 & a_1 + \\ \hline + & | & | \\ c_2 & b_1 - & \end{array} + V_{n_2}^+ \begin{array}{c|c|c} + & a_2 & a_1 + \\ \hline + & | & | \\ a_2 & c_1 - & \end{array} \right. \\ \left. + V_{n_1}^+ \begin{array}{c|c|c} + & b_2 & c_1 - \\ \hline + & | & | \\ b_2 & c_1 - & \end{array} + V_{n_1}^+ \begin{array}{c|c|c} + & b_2 & b_1 + \\ \hline + & | & | \\ b_2 & b_1 + & \end{array} \right. \\ \left. + V_{n_1}^+ \begin{array}{c|c|c} + & b_2 & b_1 + \\ \hline + & | & | \\ b_2 & b_1 + & \end{array} \right)$$

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 & = \frac{a_1^N a_2^N c_1 c_2}{b_1 b_2} \sum_{n_1 < n_2} \left( \frac{a_{21}}{b_{21}} z_1^{n_1} z_2^{n_2} + \frac{a_{12}}{b_{12}} z_1^{n_2} z_2^{n_1} \right) |n_1, n_2\rangle,
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and it is a Bethe vector. It can be checked that

$$S(z_1, z_2) = \frac{a_{12}b_{21}}{b_{12}a_{21}}.$$

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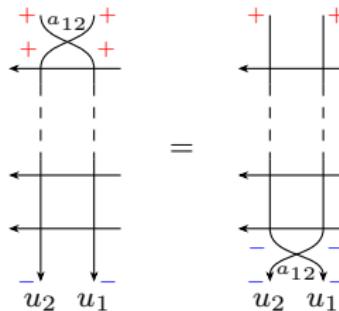
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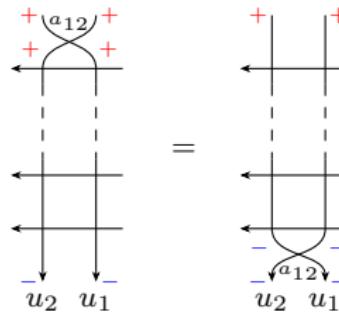
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First, the  $^{++}_{--}$ -component of this relation gives

$$B(u_1)B(u_2) = B(u_2)B(u_1). \quad (3)$$

It means that the states (2) are symmetric in  $u_i$ .

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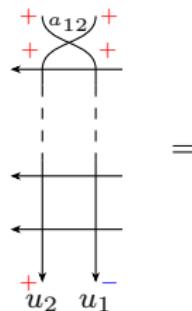
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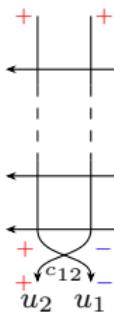
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=



+



$$= c(u_1 - u_2)B(u_2)A(u_1) + b(u_1 - u_2)A(u_2)B(u_1)$$

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$$a(u_2 - u_1)B(u_1)D(u_2) =$$

$$\begin{array}{c} \text{Diagram 1: Two vertical lines with arrows pointing left. Top line has a red '+' sign at top right. Bottom line has a blue '-' sign at top left. They intersect at points } u_1 \text{ and } u_2. \\ \text{Diagram 2: Similar to Diagram 1, but with a blue arc labeled } c_{21} \text{ above the intersection point. Top line has a red '+' sign at top right. Bottom line has a blue '-' sign at top left. They intersect at points } u_1 \text{ and } u_2. \\ \text{Diagram 3: Similar to Diagram 1, but with a blue arc labeled } b_{21} \text{ above the intersection point. Top line has a red '+' sign at top right. Bottom line has a blue '-' sign at top left. They intersect at points } u_1 \text{ and } u_2. \\ = \quad + \\ \text{Diagram 1} \quad \text{Diagram 2} \quad \text{Diagram 3} \\ = c(u_2 - u_1)B(u_2)D(u_1) + b(u_2 - u_1)D(u_2)B(u_1) \end{array}$$

# Commutation relations

To commute  $T(u)$  with  $B(u_i)$  we will need the commutations of operators  $A(u)$  and  $D(u)$  with  $B(u)$ :

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Finally we have

$$a(u_1 - u_2)B(u_1)A(u_2) = c(u_1 - u_2)B(u_2)A(u_1) + b(u_1 - u_2)A(u_2)B(u_1), \quad (4)$$

$$a(u_2 - u_1)B(u_1)D(u_2) = c(u_2 - u_1)B(u_2)D(u_1) + b(u_2 - u_1)D(u_2)B(u_1). \quad (5)$$

# Action of the transfer matrix: $n = 1$

Apply these formulas to the action of the transfer matrix.

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$$\begin{aligned} T(u)|u_1\rangle &= (A(u) + D(u))B(u_1)|\Omega_+\rangle \\ &= \left( \frac{a(u_1 - u)}{b(u_1 - u)} B(u_1)A(u) + \frac{a(u - u_1)}{b(u - u_1)} B(u_1)D(u) \right. \\ &\quad \left. - \frac{c(u_1 - u)}{b(u_1 - u)} B(u)A(u_1) - \frac{c(u - u_1)}{b(u - u_1)} B(u)D(u_1) \right) |\Omega_+\rangle. \end{aligned}$$

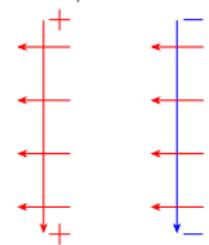
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Taking into account that  $A(u)|\Omega_+\rangle = a^N(u)|\Omega_+\rangle$ ,  $D(u)|\Omega_+\rangle = b^N(u)|\Omega_+\rangle$ .

$$A(u)|\Omega_+\rangle \quad D(u)|\Omega_+\rangle$$



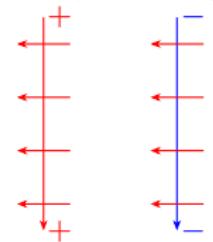
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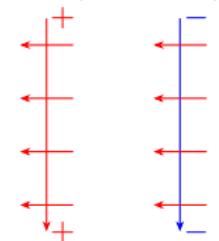
$$\begin{aligned} T(u)|u_1\rangle &= (A(u) + D(u))B(u_1)|\Omega_+\rangle \\ &= \left( \frac{a(u_1 - u)}{b(u_1 - u)} B(u_1)A(u) + \frac{a(u - u_1)}{b(u - u_1)} B(u_1)D(u) \right. \\ &\quad \left. - \frac{c(u_1 - u)}{b(u_1 - u)} B(u)A(u_1) - \frac{c(u - u_1)}{b(u - u_1)} B(u)D(u_1) \right) |\Omega_+\rangle. \end{aligned}$$

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The vector  $|u_1\rangle$  is an eigenvector of  $T(u)$  if the second term vanishes:

$$\left( \frac{b(u_1)}{a(u_1)} \right)^N = - \left( \frac{c(u_1 - u)b(u - u_1)}{c(u - u_1)b(u_1 - u)} \right) = 1 \Leftrightarrow z_1^N = 1.$$



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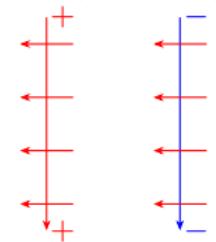
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The corresponding eigenvector is

$$\Lambda(u; u_1) = a^N(u) \frac{a(u_1 - u)}{b(u_1 - u)} + b^N(u) \frac{a(u - u_1)}{b(u - u_1)}.$$

The Takhtajan–Faddeev formulas:

$$\begin{aligned}
 A(u)|u_1, \dots, u_n\rangle &= \alpha(u; u_1, \dots, u_n)|u_1, \dots, u_n\rangle \\
 &\quad - \sum_{i=1}^n \frac{c(u_i - u)}{b(u_i - u)} \alpha(u_i; u_1, \dots, \widehat{u_i}, \dots, u_n) |u, u_1, \dots, \widehat{u_i}, \dots, u_n\rangle, \\
 D(u)|u_1, \dots, u_n\rangle &= \delta(u; u_1, \dots, u_n)|u_1, \dots, u_n\rangle \\
 &\quad - \sum_{i=1}^n \frac{c(u - u_i)}{b(u - u_i)} \delta(u_i; u_1, \dots, \widehat{u_i}, \dots, u_n) |u, u_1, \dots, \widehat{u_i}, \dots, u_n\rangle,
 \end{aligned} \tag{6}$$

where

$$\alpha(u; u_1, \dots, u_n) = a^N(u) \prod_{i=1}^n \frac{a(u_i - u)}{b(u_i - u)}, \quad \delta(u; u_1, \dots, u_n) = b^N(u) \prod_{i=1}^n \frac{a(u - u_i)}{b(u - u_i)}. \tag{7}$$

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Thus the action of the transfer matrix is

$$T(u)|u_1, \dots, u_n\rangle = \Lambda(u; u_1, \dots, u_n)|u_1, \dots, u_n\rangle + \text{unwanted terms},$$

where

$$\Lambda(u; u_1, \dots, u_n) = \alpha(u; u_1, \dots, u_n) + \delta(u; u_1, \dots, u_n)$$

# Bethe equations and transfer matrix eigenvalue

Since  $\frac{c(u)}{b(u)} = -\frac{c(-u)}{b(-u)}$ , the unwanted terms in the r.h.s. have the form

$$\frac{c(u_i - u)}{b(u_i - u)} (\alpha(u_i; u_1, \dots, \widehat{u_i}, \dots, u_n) - \delta(u_i; u_1, \dots, \widehat{u_i}, \dots, u_n)) |u, u_1, \dots, \widehat{u_i}, \dots, u_n\rangle.$$

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They vanish, if the **Bethe equations** are satisfied:

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$$\left( \frac{b(u_i)}{a(u_i)} \right)^N = \prod_{\substack{j=1 \\ (j \neq i)}}^n \frac{a(u_j - u_i)b(u_i - u_j)}{b(u_j - u_i)a(u_i - u_j)}. \quad (8)$$

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Subject to these equations the vectors  $|u_1, \dots, u_n\rangle$  are eigenvectors with the eigenvalues

$$\Lambda(u; u_1, \dots, u_n) = a^N(u) \prod_{i=1}^n \frac{a(u_i - u)}{b(u_i - u)} + b^N(u) \prod_{i=1}^n \frac{a(u - u_i)}{b(u - u_i)}. \quad (9)$$

# Explicit form of the Bethe equations

Let

$$s(u) = \begin{cases} \sin u & \text{for } |\Delta| < 1; \\ \operatorname{sh} u & \text{for } \Delta < -1; \end{cases} \quad c(u) = \begin{cases} \cos u & \text{for } |\Delta| < 1; \\ \operatorname{ch} u & \text{for } \Delta < -1. \end{cases}$$

The explicit form of the Bethe equations:

$$\left( \frac{s(u_i)}{s(\lambda - u_i)} \right)^N = \prod_{\substack{j=1 \\ (j \neq i)}}^n \frac{s(u_i - u_j + \lambda)}{s(u_i - u_j - \lambda)}. \quad (10)$$

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Let

$$u_i = \frac{\lambda}{2} + iv_i, \quad e^{ip(v)} = \frac{s(\lambda/2 + iv)}{s(\lambda/2 - iv)}, \quad e^{i\theta(v)} = \frac{s(\lambda + iv)}{s(\lambda - iv)}.$$

The variables  $v_i$  are defined in such a way that  $|z_i| = 1$  for real values of  $v_i$ .

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The variables  $v_i$  are defined in such a way that  $|z_i| = 1$  for real values of  $v_i$ . Take logarithm of the Bethe equations:

$$Np(v_i) = 2\pi I_i + \sum_{j=1}^n \theta(v_i - v_j),$$

where  $I_i \in \mathbb{Z} + \frac{1}{2}$  if  $n \in 2\mathbb{Z}$  and  $I_i \in \mathbb{Z}$  if  $n \in 2\mathbb{Z} + 1$ .