Lecture 2
Six-vertex model

A mini-course "Solvable lattice models and Bethe Ansatz"
(Ariel University, spring 2021)

Michael Lashkevich

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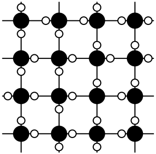
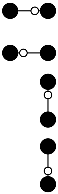
Kharkevich Institute for Information Transmission Problems

The ‘ice model’ (● is Oxygen, ∘ is Hydrogen):



Each oxygen atom has two hydrogen atom next to it.



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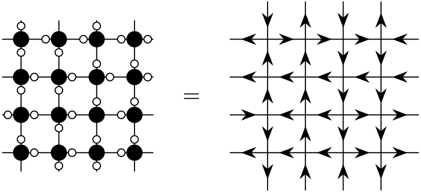

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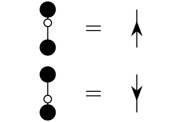

Ice model: configurations

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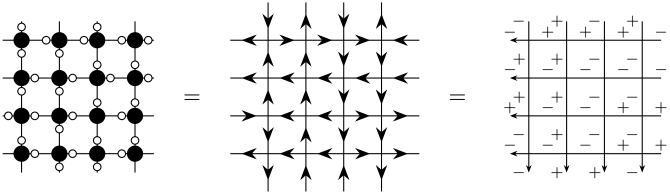
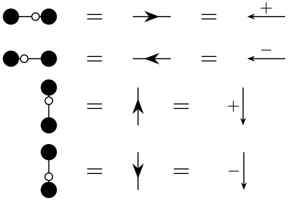


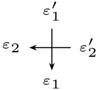
Each oxygen atom has two hydrogen atom next to it. Small arrows on the right figure define the orientation of the lattice lines and vertices, which will be important later.



Ice model: Boltzmann weights

Six-vertex model: the Boltzmann weights are associated with vertices:

$$Z = \sum_{\text{configu-rations}} \prod_{\text{vertices}} R^{\varepsilon'_1\varepsilon'_2}_{\varepsilon_1\varepsilon_2}, \quad R^{\varepsilon'_1\varepsilon'_2}_{\varepsilon_1\varepsilon_2} =$$



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Ice condition

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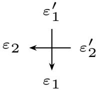


$$\boxed{\varepsilon'_1 + \varepsilon'_2 = \varepsilon_1 + \varepsilon_2}.$$

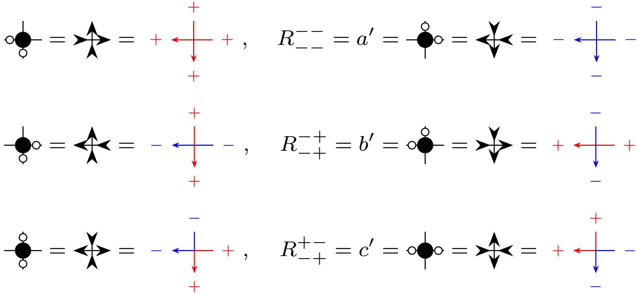
Ice condition

We have six vertex configurations

$$R^{++}_{++} = a, \qquad R^{--}_{--} = a',$$

$$R^{+-}_{+-} = b, \qquad R^{-+}_{-+} = b',$$

$$R^{-+}_{+-} = c, \qquad R^{+-}_{-+} = c'$$



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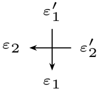
$$Z = \sum_{\substack{\text{configu-}\\\text{rations}}} \prod_{\text{vertices}} R^{\varepsilon'_1\varepsilon'_2}_{\varepsilon_1\varepsilon_2}, \qquad R^{\varepsilon'_1\varepsilon'_2}_{\varepsilon_1\varepsilon_2} =$$ 

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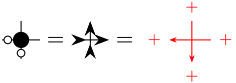
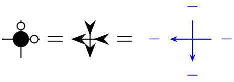
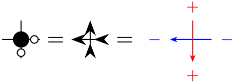
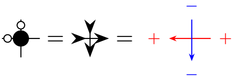
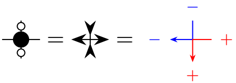
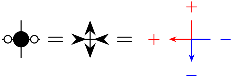
$R^{++}_{++} = a =$  , $\quad R^{--}_{--} = a' =$ 

$R^{+-}_{+-} = b =$  , $\quad R^{-+}_{-+} = b' =$ 

$R^{-+}_{+-} = c =$  , $\quad R^{+-}_{-+} = c' =$ 

$$R = \begin{pmatrix} a & & & \\ & b & c & \\ & c' & b' & \\ & & & a' \end{pmatrix} \text{ in the basis } (++), (+-), (-+), (--).$$

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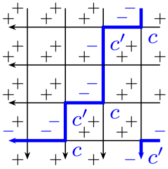

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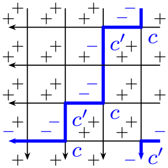

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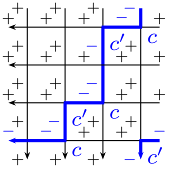

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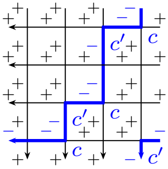

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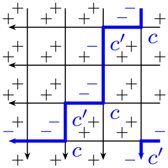

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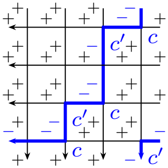

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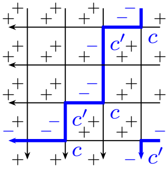

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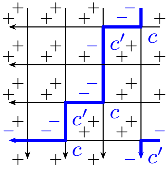

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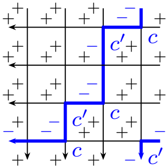

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We will consider the symmetric six-vertex model:

$$R^{-\varepsilon'_1\,-\varepsilon'_2}_{-\varepsilon_1\,-\varepsilon_2} = R^{\varepsilon'_1\varepsilon'_2}_{\varepsilon_1\varepsilon_2}$$

or

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The transfer matrix

$$T^{\varepsilon'_1\ldots\varepsilon'_N}_{\varepsilon_1\ldots\varepsilon_N} = \sum_{\mu_1\ldots\mu_N} R^{\mu_2\varepsilon'_1}_{\mu_1\varepsilon_1} R^{\mu_3\varepsilon'_2}_{\mu_2\varepsilon_2} \ldots R^{\mu_1\varepsilon'_N}_{\mu_N\varepsilon_N}. \tag{1}$$

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Let us consider the matrix R as an operator in the tensor product of two two-dimensional spaces:

$$R : \mathbb{C}^2\otimes\mathbb{C}^2 \to \mathbb{C}^2\otimes\mathbb{C}^2, \qquad v_{\varepsilon_1}\otimes v_{\varepsilon_2} \mapsto R_{\varepsilon'_1\varepsilon'_2}^{\varepsilon_1\varepsilon_2} v_{\varepsilon'_1}\otimes v_{\varepsilon'_2}.$$

Here v_ε is the natural basis in $V = \mathbb{C}^2$.

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Here v_ε is the natural basis in $V = \mathbb{C}^2$. Consider the tensor product $V_1 \otimes V_2 \otimes \cdots \otimes V_k$ of identical spaces $V_i \simeq V$. Let R_{ij} is the R matrix acting on $V_i \otimes V_j$.

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Then the transfer matrix can be written as

$$T=\mathrm{tr}_{V_0}(R_{0N}\ldots R_{02}R_{01}):\ V_1\otimes V_2\otimes\cdots\otimes V_N\to V_1\otimes V_2\otimes\cdots\otimes V_N. \tag{2}$$

The space $V_1\otimes\cdots\otimes V_N$ is called quantum space, while the space V_0 is called auxiliary space.

The operator under the trace is

$$L = R_{0N} \dots R_{02} R_{01} : \; V_0 \otimes V_1 \otimes \dots \otimes V_N \to V_0 \otimes V_1 \otimes \dots \otimes V_N . \qquad (3)$$

Six-vertex models: L operator

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We will consider it as an operator in the quantum space and a matrix in the auxiliary space

$$L = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad A, B, C, D : V_1 \otimes V_2 \otimes \cdots \otimes V_N \to V_1 \otimes V_2 \otimes \cdots \otimes V_N .$$

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Then

$$T = \mathrm{tr}_{V_0} L = A + D. \tag{4}$$

Commuting transfer matrices and Yang–Baxter equation

Integrability demands the existence of extra commuting integrals of motion I_n:

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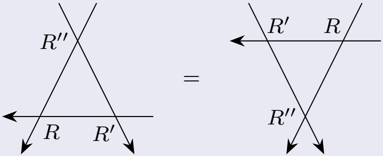
Theorem

If there exist nondegenerate matrices R', R'' such that

$$R''_{12} R'_{13} R_{23} = R_{23} R'_{13} R''_{12}, \tag{5}$$

or, graphically



(5′)

then

$$[T, T'] = 0 \tag{6}$$

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$$T'T = \tag{7}$$

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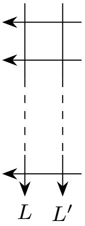

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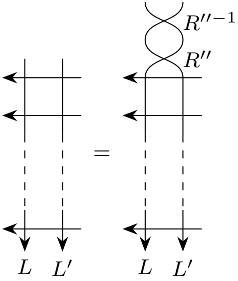

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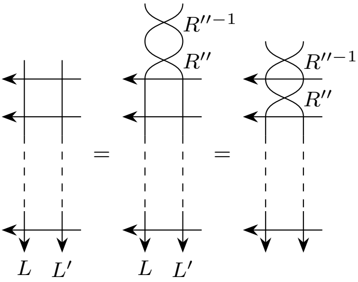

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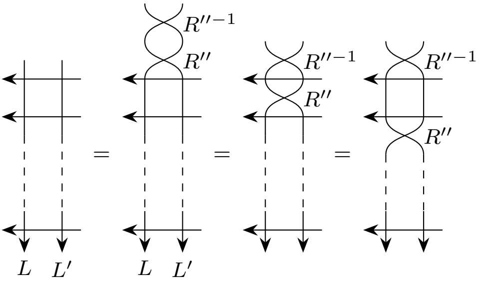

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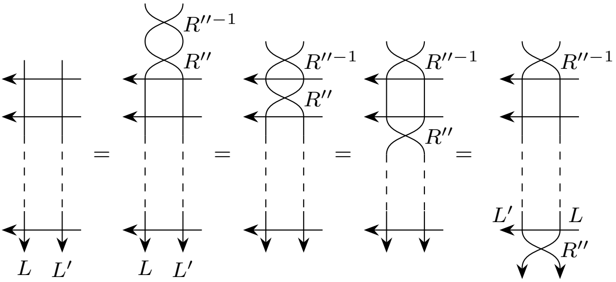

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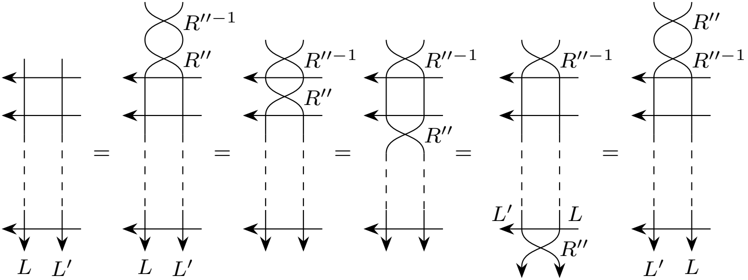

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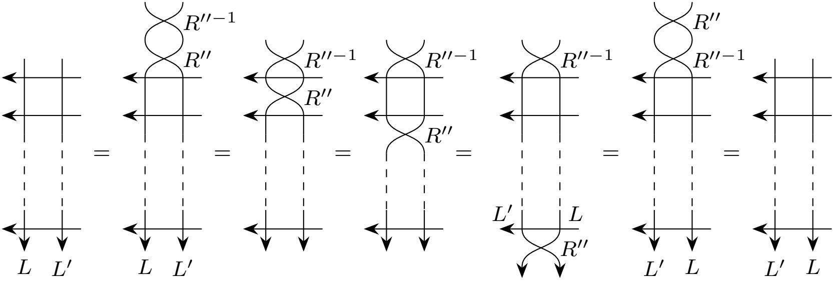

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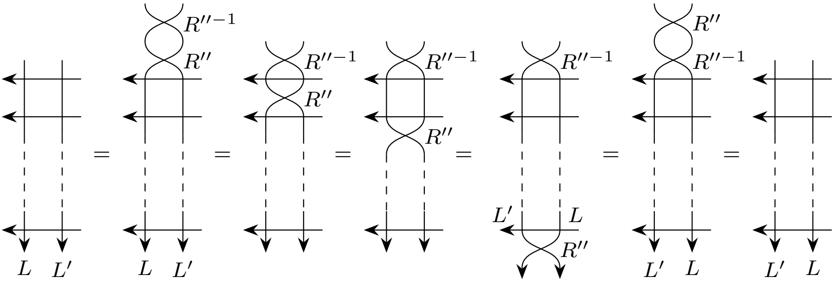

$= TT'. \quad (7)$

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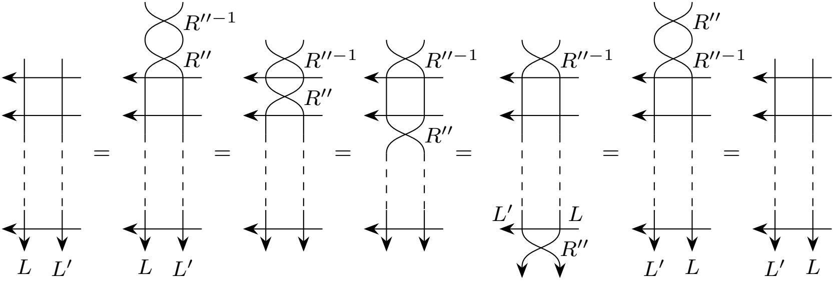

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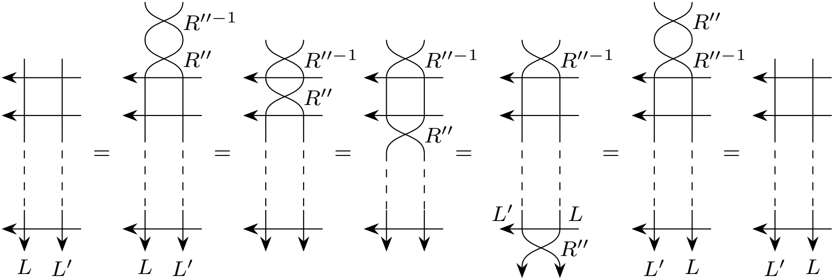

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$$R''_{12} L'_1 L_2 = L_2 L'_1 R''_{12},$$

which is proved by induction. Then

$$T'T = \mathrm{tr}_{V_1\otimes V_2}(L'_1 L_2) = \mathrm{tr}_{V_1\otimes V_2}((R''_{12})^{-1} R''_{12} L'_1 L_2) = \mathrm{tr}_{V_1\otimes V_2}((R''_{12})^{-1} L_2 L'_1 R''_{12})$$
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The solution can be found in the form

$$\begin{aligned} R &= R(\lambda, u_2 - u_3), \\ R' &= R(\lambda, u_1 - u_3), \\ R'' &= R(\lambda, u_1 - u_2) \end{aligned} \tag{8}$$

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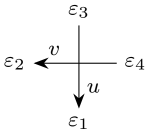
Thus we will omit the parameter λ from now on:

$$R(u) \equiv R(\lambda, u),\ \ a(u) \equiv a(\lambda, u) \text{ etc.}$$

Yang–Baxter equation: spectral parameter

The spectral parameters can be associated to lines:

$$R(\lambda, u - v)^{\varepsilon_3 \varepsilon_4}_{\varepsilon_1 \varepsilon_2} =$$



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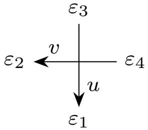

This R matrix is the solution to the Yang–Baxter equation in the form

$$\begin{aligned} R_{12}(\lambda, u_1 - u_2) R_{13}(\lambda, u_1 - u_3) R_{23}(\lambda, u_2 - u_3) \\ = R_{23}(\lambda, u_2 - u_3) R_{13}(\lambda, u_1 - u_3) R_{12}(\lambda, u_1 - u_2). \end{aligned} \tag{10}$$

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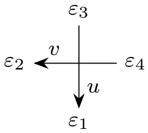

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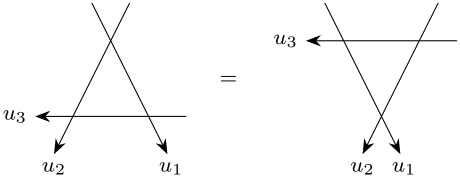

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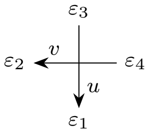

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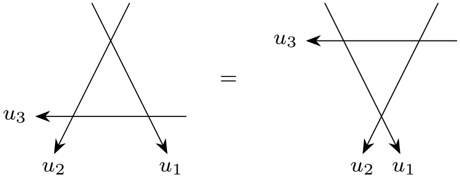

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Besides, the R matrix satisfy the relations

$$b(u)R(\lambda-u)_{\varepsilon_1\varepsilon_2}^{\varepsilon_3\varepsilon_4} = R(u)_{\varepsilon_4\ -\varepsilon_1}^{\varepsilon_2\ -\varepsilon_3}, \qquad R_{12}(u)R_{21}(-u) = 1, \qquad R(0) = P =$$

. (11)

Integrals of motion

We have

$$[T(u), T(u')] = 0 \quad \forall u, u'. \tag{12}$$

But not all the integrals of motion $T(u)$ are independent.

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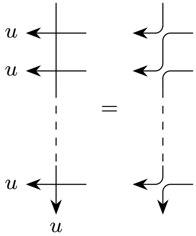

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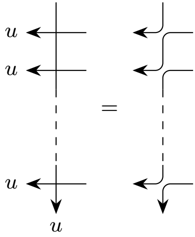

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$$T^{-1}(0)T(u) = 1 - \sum_{n=1}^{\infty} \frac{H_n u^n}{n!}. \tag{14}$$

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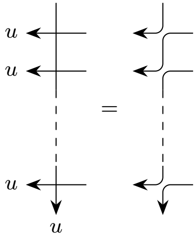
$T(0) =$ 

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$$[T(0), H_n] = [H_m, H_n] = 0 \quad \forall m, n. \tag{15}$$

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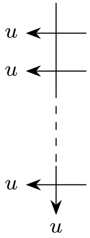
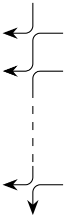
$T(0) =$  $=$  (13)

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Operators H_n are local in the sense that each of them is a sum of term, which involves a finite number $(n+1)$ of neighboring nodes.

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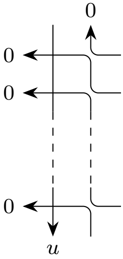

Six-vertex model and XXZ Heisenberg chain

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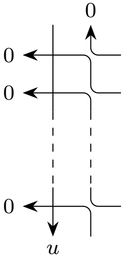

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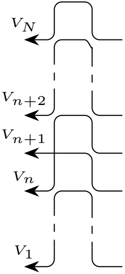

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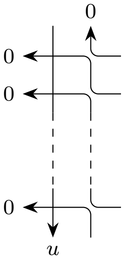

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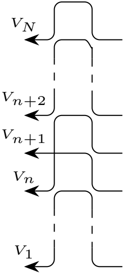

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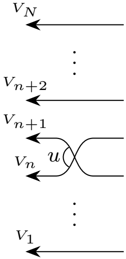

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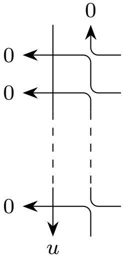

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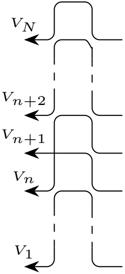

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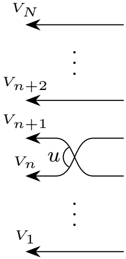

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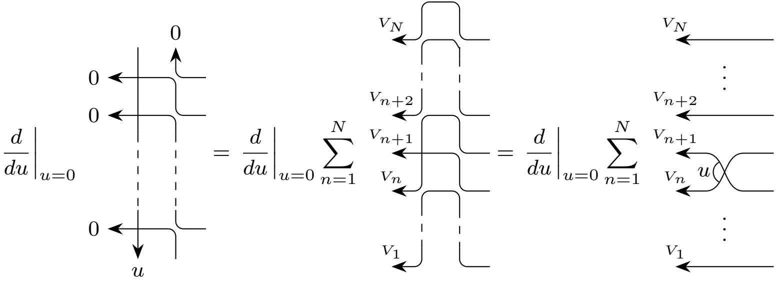

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$$\check{R}(u) = PR(u) = \begin{pmatrix} a(u) & & & \\ & c(u) & b(u) & \\ & b(u) & c(u) & \\ & & & a(u) \end{pmatrix} = 1 + \frac{u}{\sin\lambda} \begin{pmatrix} 0 & & & \\ & \cos\lambda & 1 & \\ & 1 & \cos\lambda & \\ & & & 0 \end{pmatrix} + O(u^2)$$

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This leads to the identification of the space $\mathcal{H}_N$ and the quantum space of the six-vertex model:

$$\mathcal{H}_N = \underbrace{V \otimes \cdots \otimes V}, \quad v_\pm = |\pm\rangle.$$

Due to the ice condition the z component of total spin

$$S^z = \frac{1}{2} \sum_{i=1}^{N} \sigma_n^z$$

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XXZ Heisenberg chain: pseudovacuums

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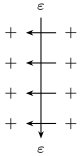

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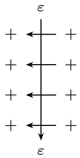

$$= (a^N(u) + b^N(u))|\Omega_+\rangle.$$

XXZ Heisenberg chain: pseudovacuums

Due to the ice condition the z component of total spin

$$S^z = \frac{1}{2}\sum_{i=1}^{N}\sigma_n^z$$

is a conserved charge:

$$[T(u), S^z] = [H_{\mathrm{XXZ}}, S^z] = 0. \tag{17}$$

Thus the space of states is split into the sum over eigenvalues of S^z.
Recall that

$$S^z|\Omega_\pm\rangle = \pm\frac{N}{2}|\Omega_\pm\rangle, \quad H_{\mathrm{XXZ}}|\Omega_\pm\rangle = -\frac{N\Delta}{2}|\Omega_\pm\rangle.$$

The action of the transfer matrix:

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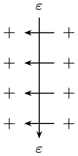
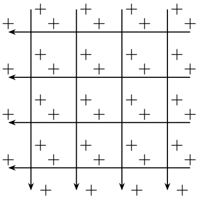


$$= (a^N(u) + b^N(u))|\Omega_+\rangle.$$

Finally,

$$T(u)|\Omega_\pm\rangle = (a^N(u) + b^N(u))|\Omega_\pm\rangle.$$

1. Ferroelectric regime: $\Delta > 0$. Let $a > b + c$. Ground configurations:



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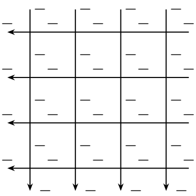

Six-vertex model: three regimes

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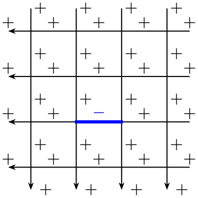

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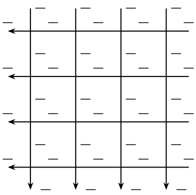

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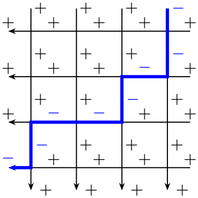

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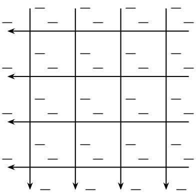

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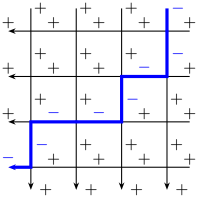

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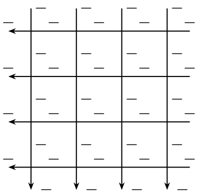

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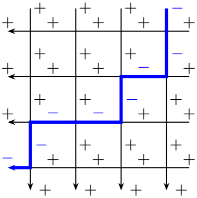

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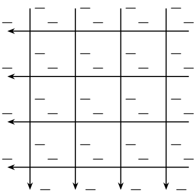

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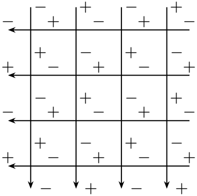

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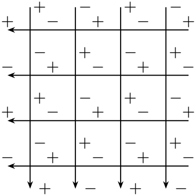

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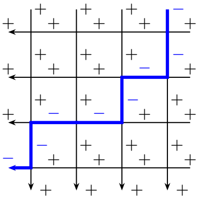

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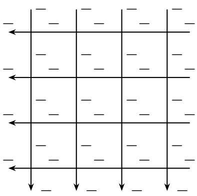

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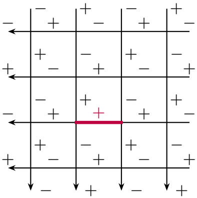

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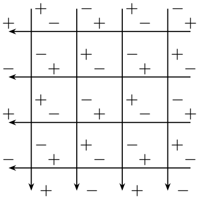

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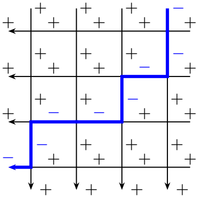

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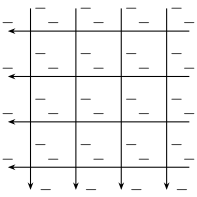

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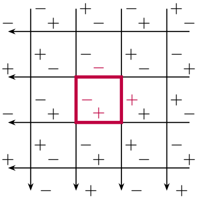

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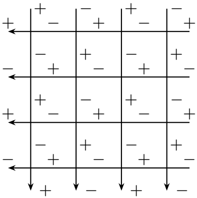

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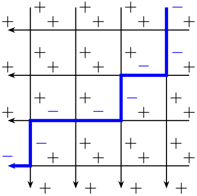

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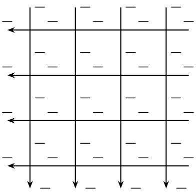

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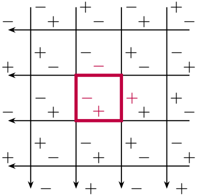

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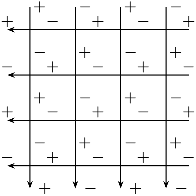

The excitations have finite weight. $\Rightarrow$ Nontrivial thermodynamics.

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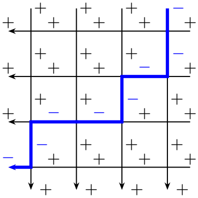

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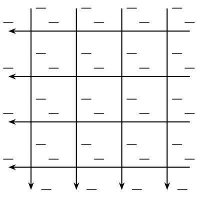

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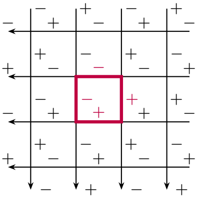

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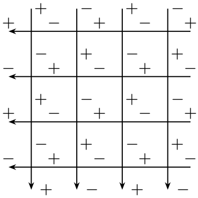

The excitations have finite weight. $\Rightarrow$ Nontrivial thermodynamics.
3. Disordered regime: $|\Delta| < 1$. No ground configurations. It turns out that this regime is always critical.