Lecture 1
Heisenberg spin chain

A mini-course "Solvable lattice models and Bethe Ansatz"
(Ariel University, spring 2021)

Michael Lashkevich

Landau Institute for Theoretical Physics,
Kharkevich Institute for Information Transmission Problems

Single spin 1/2

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It satisfies the condition

$$T\sigma_n^\alpha = \sigma_{n-1}^\alpha T. \tag{8}$$

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$$H_{\mathrm{XYZ}} = -\frac{1}{2}\sum_{n=1}^{N}\left(J_x\sigma_n^x\sigma_{n+1}^x + J_y\sigma_n^y\sigma_{n+1}^y + J_z\sigma_n^z\sigma_{n+1}^z\right) \tag{9}$$

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$$[H_{\mathrm{XYZ}}, T] = 0. \quad (10)$$

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Each term either does not change the spins or flips one spin up and another spin down. Hence

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Therefore we can split the space of states into the subspaces of spin eigenvectors:

$$\mathcal{H}_N = \bigoplus_{k=0}^{N}(\mathcal{H}_N)_k, \quad (\mathcal{H}_n)_k = \left\{|\psi\rangle \;\middle|\; S^z|\psi\rangle = (N/2-k)|\psi\rangle\right\}. \tag{14}$$

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The Hamiltonian H_{XXZ} acts as an operator on each of these subspaces.

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The space $(\mathcal{H}_N)_0$ is one-dimensional:

$$(\mathcal{H}_N)_0 = \mathbb{C}|\Omega_+\rangle, \quad |\Omega_+\rangle = |\uparrow\uparrow\ldots\uparrow\rangle. \tag{15}$$

This state can be also defined by the condition

$$\sigma_n^+|\Omega_+\rangle = 0 \quad \forall n = 1,\ldots,N. \tag{16}$$

The space $(\mathcal{H}_N)_1$ is N-dimensional:

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Eigenvectors and eigenvalues: $k = 0, 1$

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Consider the case $k = 2$. Let us search for an eigenstate in the form of a combination of wave solutions:

$$|z_1, z_2\rangle = \sum_{n_1 < n_2} (A_{12} z_1^{n_1} z_2^{n_2} + A_{21} z_2^{n_1} z_1^{n_2}) |n_1, n_2\rangle. \tag{23}$$

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These two equations determine the allowed values of the pairs (z_1, z_2). Thus, though the total energy looks additive, the sets of allowed quasimomenta of excitations are different in the cases of one-particle and of two-particle states.

Eigenvectors: Bethe Ansatz and Bethe equations

Consider general k. The Bethe Ansatz is

$$|z_1, \ldots, z_k\rangle = \sum_{n_1 < \ldots < n_k} \sum_{\sigma \in S_k} A_{\sigma_1 \ldots \sigma_k} \prod_{j=1}^{k} z_{\sigma_j}^{n_j} |n_1, \ldots, n_k\rangle.$$

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are satisfied. The eigenvalue is given by

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